From Euler's Series to Riemann's Zeta Function: The Explicit Derivation of the Analytic Continuation and the Functional Equation, with Every Step Written Out and Every Construction Drawn

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Abstract

This paper traces the path from Euler's series $1 + 2^{-s} + 3^{-s} + \cdots$ of 1737 to Riemann's zeta function of 1859, and it does so at the finest grain: every letter is named the first time it appears, every displayed equation is followed by the single operation that turns it into the next, every technical word is defined before it is used, and every person named is placed in time relative to Riemann's paper. Euler had the series and its product over the primes for real exponents greater than 1, the values of the series at the even integers, and in 1749 a conjectured relation between the series at s and the series at $1 - s$. That relation could not be a theorem, because for $s > 1$ the series at $1 - s$ is a sum of growing numbers that settles to no value. Riemann gave the symbol $\zeta(s)$ a definite meaning at every complex number s other than 1 and proved the relation. His procedure consisted of three operations on one integral. First, he rewrote each term n^{-s} as an integral of Euler's Gamma type, which moved the variable s out of the series and into the single factor x^{s-1} of an integrand. Second, he replaced the path of integration from 0 to ∞ by a loop that comes in from $+\infty$, circles the origin once, and returns to $+\infty$; this turned the integral into a function with a derivative at every complex s, because the loop never meets the one point, $x = 0$, at which convergence had depended on s. Third, he pushed that loop outward onto the points $x = \pm 2\pi i n$ where $1/(e^x - 1)$ becomes infinite, and the residues at those points produced the functional equation $\zeta(1 - s) = 2(2\pi)^{-s} \cos(\pi s/2)\, \Gamma(s)\, \zeta(s)$. He then gave a second derivation from Jacobi's transformation of the theta series, which yields the symmetric form $\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1 - s)/2)\zeta(1 - s)$ and the function $\xi(t)$ whose zeros are the subject of his hypothesis. Every construction that lives on a line or in a plane is drawn, with each symbol labelled. A section on Riemann's education and library records which books, lectures and journals supplied him with each tool, a section explains at what single point the primes enter the calculus and by what device they come out, and a closing section settles which functions can select the primes (an elementary one, a polynomial whose positive values are the primes, and one built from ζ), which cannot (a polynomial all of whose values are prime), and states the open problem of the least cost of such a function.

1 Introduction: what is answered, and how the paper is written

The question answered is operational. Which equations did Riemann start from, what did he do to them, and in what order, so that the series

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots,$$

which settles to a value only when the exponent s is a real number greater than 1 (or a complex number whose real part is greater than 1), became a function with a definite value at every complex number s except $s = 1$.

The paper is written under four rules, each a response to the way mathematical prose usually fails its reader.

1. **Every letter is named when it first appears.** The sentence that introduces a letter says what kind of thing it stands for and what values it takes. Thus: n is a positive whole number, one of $1, 2, 3, \ldots$; s is the exponent, a real number greater than 1 until Section 8, where it becomes a complex number; p stands for a prime number, one of $2, 3, 5, 7, 11, \ldots$, and a sum "over p" means a sum with one term for each prime. When a letter is reused later with the same meaning, the meaning is restated at the point of reuse.

2. **Every step is one operation, and the operation is named.** When one displayed line becomes the next, the text between them says which operation produced the change: a substitution (and what was substituted for what), an integration by parts (and which factor was differentiated), a limit (and of which quantity), a comparison of two integrands (and why one is larger). The reader is never asked to reconstruct a step.

3. **Every technical word is defined before it is used, in words a reader can check.** Section 4 collects the definitions; the first time a word is needed the text repeats the definition in a sentence rather than pointing at a number.

4. **Every person named is placed in time.** Section 3 lists each person with dates and with what of theirs this paper uses, before or after Riemann's paper of November 1859. Section 2 says which of these things were physically in Riemann's hands and how they got there.

5. **Every construction that lives on a line or in a plane is drawn.** A path of integration, a cut, a region, a step function, a comparison of a sum with an integral: each is drawn, and the caption says what every symbol in the drawing stands for and where the drawing is used.

The answer rests on five documents, listed here with what each contributes.

1. Euler, *Variae observationes circa series infinitas* (presented to the St Petersburg Academy in 1737, printed in 1744): the series above for real $s > 1$, its rewriting as a product with one factor for each prime, and the conclusion that the sum of the reciprocals of the primes, $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$, grows beyond every bound.

2. Euler, *Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques* (read to the Berlin Academy in 1749, printed in 1768): the conjectured relation between the series at s and at $1 - s$, checked at the whole numbers by attaching values to series that do not converge.

3. Jacobi, *Fundamenta nova theoriae functionum ellipticarum* (Königsberg, 1829), page 184: the identity $\theta(1/x) = \sqrt{x}\,\theta(x)$ for the series $\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x}$, which Riemann cites by page.

4. Riemann, *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (doctoral dissertation, Göttingen, 1851): the theory of functions of a complex variable that have a derivative, including the fact that such a function on a connected region is fixed by its values on any small piece of the region.

5. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (Monatsberichte der Berliner Akademie, November 1859): the eight-page paper in which all of the following is done. Every quotation of Riemann below is from the English translation by D. R. Wilkins (1998).

2 What Riemann had in his hands: his education, his books and his journals

Nothing in the 1859 paper is invented from nothing. Each tool came from a book Riemann owned or borrowed, a lecture he sat in, or a journal that arrived at the Göttingen library. This section records where each came from. The biographical facts are from Dedekind's memoir of Riemann (printed in Riemann's collected works, 1876) and from the collected works themselves.

Schooling, 1840–1846. Riemann was born in 1826 in Breselenz, in the kingdom of Hanover. He attended the gymnasium in Hanover from 1840 and the Johanneum in Lüneburg from 1842 to 1846. Dedekind records that the director of the Johanneum, Schmalfuss, lent him Legendre's *Théorie des nombres*, an 859-page treatise on the theory of whole numbers, and that Riemann returned it after six days saying he had worked through it. That book contains Legendre's guess, from tables, that the number of primes below x is close to $x/(\log x - 1.08366)$. So the problem of the 1859 paper, the number of primes below a given bound, was in Riemann's hands at the age of about sixteen, in the form of an empirical formula with no proof.

Göttingen, 1846–1847: Gauss. Riemann entered the University of Göttingen in 1846, at first to study theology, and moved to mathematics within the year. He attended lectures of Gauss (1777–1855) on the method of least squares and lectures of Moritz Stern on definite integrals and numerical methods. From Gauss's printed work he had the *Disquisitiones Arithmeticae* (1801) and the 1812 memoir on the hypergeometric series, *Disquisitiones generales circa seriem infinitam*, in which Gauss introduces the function $\Pi(z)$, equal to $z!$ at the whole numbers, proves its basic identities, and proves the multiplication formula of which Legendre's duplication formula is the case of two factors. Riemann's notation $\Pi(s-1)$ in 1859 is Gauss's notation from this memoir, and "the known properties of the function Π" that Riemann invokes for the symmetric form of the functional equation are the identities proved there.

Berlin, 1847–1849: Dirichlet, Jacobi, Eisenstein. Riemann spent two years at the University of Berlin, where he attended the lectures of Dirichlet (1805–1859) on number theory and on definite integrals, of Jacobi (1804–1851) on analytical mechanics and higher algebra, and of Steiner on geometry, and became a friend of Eisenstein (1823–1852), who lectured on elliptic functions. Three things entered his hands here.

- Dirichlet's 1837 paper proving that every arithmetic progression $a, a+m, a+2m, \dots$ with a and m having no common factor contains infinitely many primes. That paper takes Euler's series and product, multiplies the terms by a periodic sign pattern, keeps the exponent s real and greater than 1, and studies what happens as s decreases to 1. It is the direct model for the opening of Riemann's paper, which keeps Dirichlet's starting point and changes one thing: s is allowed to be complex. Riemann names Gauss and Dirichlet in his first paragraph as the two who had long been occupied with the subject.

- Jacobi's *Fundamenta nova* (1829) and his lectures: the theory of elliptic functions and of the theta series, including the transformation identity that Riemann cites by page number.

- Eisenstein's method. The historian André Weil reported in 1989 that Eisenstein's own copy of Gauss's *Disquisitiones* contains, in Eisenstein's hand and dated 1849, a proof of a functional equation for a series of Dirichlet's type by means of the Poisson summation formula, which is the method of Section 14 below. Riemann and Eisenstein were in daily contact in Berlin in 1849. The documents do not show whether Riemann saw the note; they show that the method was in the air among the people he talked to.

Cauchy's calculus of residues, in print since 1825. The tool that moves contours in Sections 11 and 13 is Cauchy's. Cauchy (1789–1857) published the integral theorem in a memoir of 1825, *Mémoire sur les intégrales définies, prises entre des limites imaginaires*, and the calculus of residues in his *Exercices de mathématiques* between 1826 and 1830, all in French and all in the Göttingen library by Riemann's time. Riemann's 1851 dissertation rebuilt this theory on a new foundation (the pair of partial differential equations now called the Cauchy–Riemann equations) and was examined by Gauss. The three facts of Section 8 are the content of that dissertation, in Riemann's own words.

Fourier's theorem, worked over in 1853–1854. Riemann's Habilitation thesis of 1854, *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, is a historical and mathematical study of Fourier's series, with Fourier's *Théorie analytique de la chaleur* (1822) and Dirichlet's 1829 convergence proof as its sources. So when the 1859 paper says "by Fourier's theorem" for the inversion that recovers the prime-counting function from $\log \zeta$, it refers to a theorem Riemann had spent a year on.

Euler's works, in print since the 1740s. Euler's *Introductio in analysin infinitorum* (1748) contains the product over the primes and the values $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and so on; his *Institutiones calculi differentialis* (1755) contains the Bernoulli numbers and the summation formula now named for Euler and Maclaurin, which gives Stirling's formula; the Berlin and St Petersburg academy volumes contain the 1737 and 1749 papers. Riemann cites Euler only for the product. Whether he knew the 1749 paper, where Euler guesses the functional equation, is not decided by any document.

Chebyshev's two papers, 1848 and 1852. Chebyshev (1821–1894) published, in the St Petersburg academy memoirs for 1848 and in Liouville's Paris journal for 1852, the first proved bounds on the number of primes below x: between 0.92 and 1.11 times $x/\log x$ for large x. The first of these papers uses Euler's series with a real exponent $s > 1$ and its behaviour as s decreases to 1. Riemann does not cite Chebyshev; the Paris journal was taken by Göttingen, and historians (Edwards, 1974, Chapter 1) judge it likely that he had read the work.

The journals. The German mathematical journal, Crelle's *Journal für die reine und angewandte Mathematik* (Berlin, from 1826), carried Abel, Jacobi, Dirichlet, Eisenstein and Kummer; Liouville's *Journal de mathématiques pures et appliquées* (Paris, from 1836) carried Cauchy's school and Chebyshev; the *Comptes rendus* of the Paris Academy (from 1835) carried Cauchy's weekly notes; and the proceedings of the Berlin and St Petersburg academies carried Dirichlet, Euler and Chebyshev. All were held at Göttingen.

The occasion of the paper. Gauss died in 1855 and Dirichlet took his chair; Riemann, who had been lecturing at Göttingen since 1854, became Dirichlet's colleague and was promoted in 1857. Dirichlet died in May 1859 and Riemann succeeded to the chair in July. In August 1859 the Berlin Academy elected him a corresponding member, and the eight-page paper is the report he sent the Academy in return, presented in November 1859. Its stated goal is a formula for the number of primes below a given bound; the analytic continuation of ζ is the instrument built on the way.

3 The people, in order of time

Every person named in this paper is listed here by date of birth, with the one thing of theirs that the paper uses and with its position relative to Riemann's paper of November 1859. Several results are called by the names of people who worked after Riemann; in each such case the table says so, and the text says what Riemann himself had in place of the later name.

Person	Lived	What of theirs this paper uses	Relative to 1859
Jakob Bernoulli	1655–1705	The Bernoulli numbers B_k (*Ars Conjectandi*, 1713), which give the values of ζ at the negative whole numbers.	Before.
Christian Goldbach	1690–1764	The observation, in a letter to Euler of 1752, that no polynomial can take only prime values (Theorem 17.1).	Before.
James Stirling	1692–1770	Stirling's formula for $\log\Gamma(z)$ at large z (1730), which supplies the main term in the count of zeros.	Before.
Leonhard Euler	1707–1783	The Gamma function (1729); $\zeta(2) = \pi^2/6$ and the even values (1735–1740); the series and the product over the primes (1737); the relation between $\eta(s)$ and $\eta(1-s)$ at the whole numbers (1749); the reflection formula for Γ (1771).	Before.
Joseph-Louis Lagrange	1736–1813	The first proof (1771) of Wilson's theorem, Theorem 17.2.	Before.
John Wilson	1741–1793	The theorem that n is prime exactly when n divides $(n-1)!+1$ (stated 1770), the basis of the elementary selector of Section 17.	Before.
Adrien-Marie Legendre	1752–1833	The duplication formula for Γ (1809), the letter Γ itself (1811), and the *Théorie des nombres* that Riemann read at sixteen.	Before.
Joseph Fourier	1768–1830	The Fourier integral and its inversion formula (1811; *Théorie analytique de la chaleur*, 1822), which Riemann calls "Fourier's theorem" and uses to pass from $\log\zeta$ back to the primes. Fourier was nine years older than Gauss.	Before.
Carl Friedrich Gauss	1777–1855	The notation $\Pi(s)$ for the factorial function and its identities (1812), which Riemann uses; the guess from tables that the number of primes below x is close to $\mathrm{Li}(x)$ (made 1792–93, reported in a letter to Encke in 1849). Gauss examined Riemann's dissertation and died four years before the paper.	Before.
Siméon Denis Poisson	1781–1840	The summation formula $\sum_n f(n) = \sum_m \hat{f}(m)$ (1823), by which Jacobi's theta identity is proved in Section 14.	Before.
Augustin-Louis Cauchy	1789–1857	The integral theorem (1825) and the calculus of residues (1826–1830): the tools by which contours are moved in Sections 11 and 13.	Before.
August Ferdinand Möbius	1790–1868	The function $\mu(n)$ and the inversion rule named after him (1832), by which Riemann recovers $\pi(x)$ from $J(x)$ in Section 16.	Before; alive in 1859.
Niels Henrik Abel	1802–1829	The 1826 theorem on power series that justifies the summation rule now called Abel summation; Euler had used the rule in 1749, before Abel was born.	Before.
Carl Gustav Jacob Jacobi	1804–1851	The transformation $\theta(1/x) = \sqrt{x}\,\theta(x)$ of the theta series (*Fundamenta nova*, 1829, p. 184), cited by Riemann; Riemann attended his lectures.	Before.

Person	Lived	What of theirs this paper uses	Relative to 1859
Peter Gustav Lejeune Dirichlet	1805–1859	The 1837 paper on primes in arithmetic progressions, the model for Riemann's opening; Riemann's teacher in Berlin and his colleague at Göttingen.	Before; died six months before the paper.
Karl Weierstrass	1815–1897	The comparison test for uniform convergence and the theorem that a uniform limit of functions with derivatives again has a derivative (lectures from the 1840s, printed 1880). Used here as the modern name for a justification Riemann drew from Cauchy.	Contemporary.
Pafnuty Chebyshev	1821–1894	The first proved bounds on the number of primes below x (1848, 1852), using Euler's series with real s. Not cited by Riemann.	Before.
Gotthold Eisenstein	1823–1852	A manuscript proof (1849) of a functional equation by Poisson summation, found by Weil in 1989; Riemann's friend in Berlin.	Before.
Bernhard Riemann	1826–1866	The dissertation (1851) and the paper (1859) that this paper is about.	—
Hermann Hankel	1839–1873	Applied the loop contour of Section 11 to Γ itself (1864); hence the name "Hankel contour" for a path Riemann used five years earlier.	After.
James Whitbread Lee Glaisher	1848–1928	The identity $P(s) = \sum_k \mu(k) \log \zeta(ks)/k$ for the sum over primes (1891), Lemma 17.3.	After.
Hjalmar Mellin	1854–1933	Systematised the transform $f \mapsto \int_0^\infty f(x) x^{s-1} dx$ and its inversion (1890s); the name "Mellin transform" for an operation Riemann performed in 1859.	After.
Hans von Mangoldt	1854–1925	Proved Riemann's count $N(T)$ of the zeros and the explicit formula for $J(x)$ (1895).	After.
Giacinto Morera	1856–1909	Morera's theorem (1886), the converse of Cauchy's integral theorem; used here as the modern name for the argument that an integral depending on a parameter has a derivative in that parameter.	After.
Jacques Hadamard	1865–1963	The theory of entire functions of finite order and the product formula for ξ (1893); the prime number theorem (1896). Born six years after the paper.	After.
Charles-Jean de la Vallée Poussin	1866–1962	The prime number theorem (1896), independently of Hadamard.	After.
Edmund Landau	1877–1938	The notation $\pi(x)$ for the number of primes not exceeding x (1909); with Walfisz, the natural boundary of $P(s)$.	After.
Arnold Walfisz	1892–1962	With Landau (1920): the prime zeta function $P(s)$ has the line $\operatorname{Re} s = 0$ as a natural boundary.	After.
Carl Pomerance	born 1944	With Lenstra: the improved cost bound for the Agrawal–Kayal–Saxena selector.	After.
Yuri Matiyasevich	born 1947	The theorem (1970) that every machine-listable set of whole numbers is the set of positive values of a polynomial; the ten-variable prime polynomial (1977).	After.
Hendrik Lenstra	born 1949	With Pomerance: the improved cost bound for the Agrawal–Kayal–Saxena selector.	After.

Person	Lived	What of theirs this paper uses	Relative to 1859
C. P. Willans	(paper of 1964)	The elementary selector $\lfloor \cos^2(\pi((n-1)!+1)/n) \rfloor$ and a formula for the n-th prime.	After.
Manindra Agrawal	born 1966	With Kayal and Saxena (2002): the first selector of polynomial cost.	After.
James P. Jones, Daihachiro Sato, Hideo Wada, Douglas Wiens	(paper of 1976)	The explicit polynomial in 26 variables whose positive values are exactly the primes.	After.
Neeraj Kayal	born 1979	With Agrawal and Saxena (2002), as above.	After.
Nitin Saxena	born 1981	With Agrawal and Kayal (2002), as above.	After.

4 The terms used, each defined in checkable words

Each term below is defined so that a reader can test, for a given object, whether the term applies. The definitions are used in the order in which the paper needs them. A complex number is written $z = u + iv$ with u and v real; $|z| = \sqrt{u^2 + v^2}$ is its modulus, the distance from 0 to z in the plane.

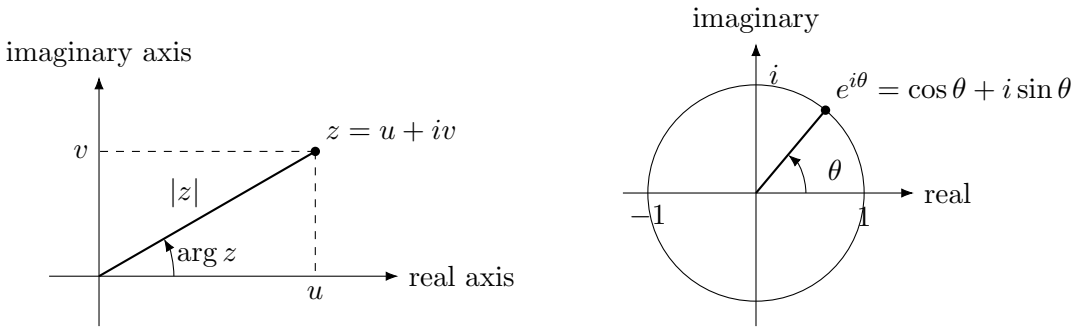


Figure 1: Left: a complex number $z = u + iv$ drawn as a point of the plane. u is its real part (horizontal coordinate), v its imaginary part (vertical coordinate), $|z| = \sqrt{u^2 + v^2}$ its modulus (the distance from 0), and $\arg z$ its argument (the angle from the positive real axis, counterclockwise positive). Right: the unit circle. The point $e^{i\theta} = \cos\theta + i\sin\theta$ lies on the circle at angle θ, so $|e^{i\theta}| = 1$; every nonzero z is $|z|e^{i \arg z}$.

4.1 Limits, convergence and divergence

1. **Limit of a sequence.** A sequence of numbers $a_1, a_2, a_3, \ldots$ has the limit L if, for every positive tolerance ε, there is a stage N beyond which every term satisfies $|a_n - L| < \varepsilon$. Written $a_n \to L$. Example: $a_n = 1/n$ has limit 0, because for tolerance ε every $n > 1/\varepsilon$ gives $|1/n - 0| < \varepsilon$.

2. **Convergent series; the sum of a series.** A series $a_1 + a_2 + a_3 + \cdots$ is a sequence in disguise: its partial sums are $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, and so on. The series converges if the sequence $S_1, S_2, S_3, \ldots$ has a limit, and that limit is called the sum of the series, written $\sum_{n=1}^{\infty} a_n$. If the partial sums have no limit, the series diverges, and the symbol $\sum_{n=1}^{\infty} a_n$ is given no meaning. The test used throughout for series of positive terms: the partial sums then increase, and an increasing sequence has a limit exactly when it is bounded (it cannot go up forever below a ceiling without settling); so a series of positive terms converges exactly when its partial sums stay below some fixed number, and diverges exactly when its partial sums eventually exceed every fixed number.

3. **Absolute convergence.** A series $\sum a_n$ of real or complex terms converges absolutely if the series of moduli $\sum |a_n|$ converges. An absolutely convergent series converges, and its terms

may be added in any order, grouped, or split into two sub-series, with the same total; these are the only manipulations of infinite series this paper performs, and absolute convergence is checked each time.

4. **Limit of a function at a point; continuity.** $f(z) \to L$ as $z \to a$ means: for every tolerance $\varepsilon > 0$ there is a radius $\rho > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - a| < \rho$. The function is continuous at a if $f(z) \to f(a)$ as $z \to a$.

5. **Improper integral.** $\int_0^\infty g(x)\,dx$ means the limit of $\int_a^b g(x)\,dx$ as a decreases to 0 and b increases to ∞, taken separately; the integral converges if both limits exist. For positive g, convergence at the end $b \to \infty$ is tested by comparing g with a known integrable function such as e^{-x} or x^{-2}, and at the end $a \to 0$ by comparing with x^c, whose integral over $(0, 1)$ is $1/(c + 1)$ when $c > -1$ and infinite when $c \le -1$.

4.2 Regions, derivatives, and the word "analytic"

6. **Open set; connected set; region.** A set D in the complex plane is open if around every point of D there is a small disc lying entirely in D; an open set has no edge points of its own. It is connected if any two of its points can be joined by a path that stays inside D. A region is an open connected set. Examples used: the half-plane of all s with $\operatorname{Re} s > 1$ is a region (the disc of radius $\operatorname{Re} s - 1$ around any point of it lies inside it, and any two points are joined by the straight segment between them); the whole plane with the single point 1 removed is a region; the plane with the ray $x \ge 0$ removed is a region. The closed half-plane $\operatorname{Re} s \ge 1$ is not open, because a point on the line $\operatorname{Re} s = 1$ has no disc around it inside the set.

7. **Complex derivative.** A function f defined near a point a has a complex derivative at a if the quotient $\dfrac{f(a + h) - f(a)}{h}$ tends to one limit as the complex number h tends to 0, however h approaches 0 (along the real axis, along the imaginary axis, along a spiral). The limit is written $f'(a)$. The condition is stronger than the real-variable one, because h may approach from every direction. Example of success: $f(z) = z^2$; then $f(a + h) - f(a) = 2ah + h^2$, the quotient is $2a + h$, and this tends to $2a$ from every direction, so $f'(a) = 2a$. Example of failure: $f(z) = \bar{z}$, the mirror image of z in the real axis; then $f(a + h) - f(a) = \bar{h}$, the quotient is $\bar{h}/h$, which equals 1 when h is real and -1 when h is purely imaginary, so no single limit exists and $\bar{z}$ has no complex derivative anywhere.

8. **Analytic function on a region.** A function is analytic on a region D if it has a complex derivative at every point of D. The word is a test with a yes-or-no answer at each point: polynomials pass everywhere; e^z, $\sin z$, $\cos z$ pass everywhere; $1/(z - 1)$ passes at every point except $z = 1$, where it is not defined; $\bar{z}$ and $|z|$ fail at every point; the series $\sum n^{-s}$ passes at every s with $\operatorname{Re} s > 1$ (proved in Section 8). Cauchy (1825–1831) and Riemann (1851) proved that an analytic function on a region is, near each of its points a, the sum of a convergent power series $c_0 + c_1(z - a) + c_2(z - a)^2 + \cdots$, and therefore has derivatives of every order; this paper uses that fact and the two consequences that follow.

9. **Entire function.** A function analytic on the whole plane: polynomials, e^z, $\sin z$, $1/\Gamma(z)$, and (Section 11) the loop integral $I(s)$ and Riemann's $\xi(t)$.

10. **Cauchy's integral theorem (1825).** If f is analytic on a region D and γ is a closed path inside D that can be shrunk to a point without leaving D, then $\int_\gamma f(z)\,dz = 0$. Consequence used repeatedly: if two paths from A to B together bound a piece of D, the integrals of f along them are equal, so a path may be moved through D without changing the integral.

11. **Uniqueness of continuation (Riemann, 1851).** If f and g are analytic on a region D and agree on a set of points having a limit point inside D, then $f = g$ everywhere on D. Proved in Section 8 as Fact 2. Consequence: a formula that is analytic on a larger region and agrees with $\sum n^{-s}$ on $\operatorname{Re} s > 1$ is *the* continuation, not *a* continuation.

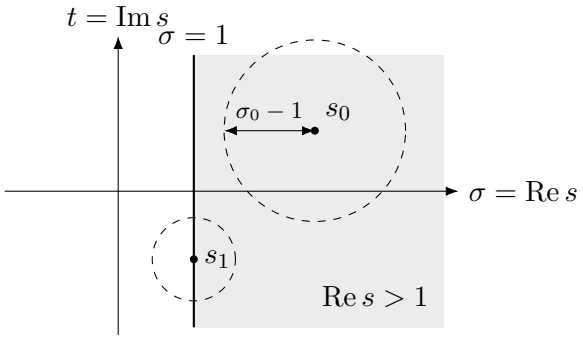


Figure 2: The half-plane $\operatorname{Re} s > 1$ (shaded) is a region. Around any of its points $s_0 = \sigma_0 + it_0$ the disc of radius $\sigma_0 - 1$ (dashed) lies inside it, so the set is open; any two of its points are joined by the straight segment between them, so it is connected. The closed half-plane $\operatorname{Re} s \geq 1$ is not open: every disc around the boundary point s_1 on the line $\sigma = 1$ sticks out to the left of the line.

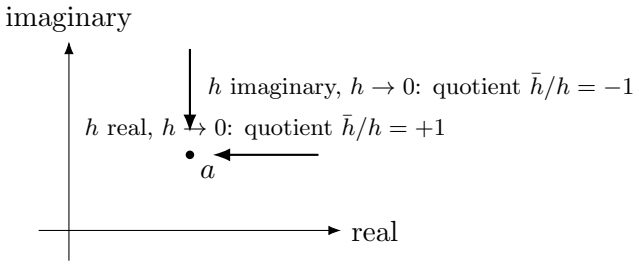


Figure 3: The test for a complex derivative at the point a: the quotient $\big(f(a+h) - f(a)\big)/h$ must tend to the same limit whichever way the complex increment h shrinks to 0. For $f(z) = \bar{z}$ (reflection in the real axis) the quotient is $\bar{h}/h$, which equals $+1$ on the horizontal approach and -1 on the vertical one; there is no single limit, so $\bar{z}$ has no complex derivative at a.

4.3 Singularities, poles, residues

12. **Isolated singularity; pole; order of a pole; simple pole; meromorphic.** A point a is an isolated singularity of f if f is analytic on a small disc around a with the centre a removed. It is a pole of order m if $f(z) = g(z)/(z-a)^m$ near a with g analytic at a and $g(a) \neq 0$; a pole of order 1 is called simple. A function is meromorphic on a region if it is analytic there except at isolated poles. Examples: $1/(e^z - 1)$ has simple poles at $z = 2\pi i k$ for every whole number k and is analytic elsewhere; $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \ldots$; the continued $\zeta(s)$ has one simple pole, at $s = 1$.

13. **Laurent expansion; residue.** Near an isolated singularity a a function is the sum of a series in positive and negative powers, $f(z) = \sum_{k=-m}^{\infty} c_k (z-a)^k$ (Laurent, 1843; for a pole of order m the negative powers stop at $k = -m$). The residue of f at a, written $\operatorname{Res}_{z=a} f$, is the one coefficient c_{-1}, the coefficient of $1/(z-a)$. Why that coefficient: integrate the expansion term by term once counterclockwise around a small circle centred at a. For $k \neq -1$ the function $(z-a)^k$ has the antiderivative $(z-a)^{k+1}/(k+1)$, which returns to its starting value after one circuit, so its integral is 0. For $k = -1$, put $z - a = re^{i\theta}$ with θ from 0 to 2π; then $dz = ire^{i\theta} d\theta$ and $\frac{dz}{z-a} = i\, d\theta$, whose integral is $2\pi i$. Hence $\oint f(z)\, dz = 2\pi i\, c_{-1}$. Two rules for computing residues: if a is a simple pole, $\operatorname{Res}_{z=a} f = \lim_{z \to a} (z-a) f(z)$; and if $f = g/h$ with g analytic at a, $h(a) = 0$ and $h'(a) \neq 0$, then $\operatorname{Res}_{z=a} f = g(a)/h'(a)$, because $h(z) = h'(a)(z-a) + (\text{smaller})$ so $(z-a)f(z) \to g(a)/h'(a)$.

14. **The residue theorem (Cauchy, 1825–1831).** If f is analytic inside and on a closed path γ run counterclockwise, except at finitely many isolated singularities $a_1, \ldots, a_n$ inside γ, then $\oint_\gamma f(z)\, dz = 2\pi i \,(\operatorname{Res}_{a_1} f + \cdots + \operatorname{Res}_{a_n} f)$. Reason: cut small discs around the a_j out of the

inside of γ; on what is left f is analytic, so by Cauchy's theorem the integral around γ equals the sum of the integrals around the small circles, each of which is $2\pi i$ times a residue by the previous item.

15. **Path; orientation; positive and negative sense.** A path is a curve $z(\tau)$ for τ running over an interval $[\alpha, \beta]$, made of finitely many smooth pieces, and $\int_\gamma f(z)\,dz$ means $\int_\alpha^\beta f(z(\tau))\,z'(\tau)\,d\tau$. Running the path backwards changes the sign of the integral. A closed path is run in the positive sense when the enclosed region lies to the left of the direction of travel, which for a circle means counterclockwise; the negative sense is clockwise. These are Riemann's words "in a positive sense" and "in a negative sense".

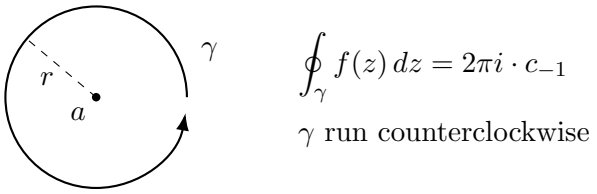


Figure 4: The residue at an isolated singularity a. Near a, $f(z) = \sum_{k \geq -m} c_k (z-a)^k$; integrating around the small circle γ of radius r once counterclockwise, every term with $k \neq -1$ returns to its starting value and contributes 0, and the term $c_{-1}/(z-a)$ contributes $2\pi i\, c_{-1}$. The residue $\mathrm{Res}_{z=a} f$ is this c_{-1}.

4.4 Many-valued powers, branches, cuts

16. **Argument; logarithm of a complex number; branch; branch cut.** Every nonzero complex number z can be written $z = |z|e^{i\theta}$, and the angle θ is called an argument of z, written $\arg z$; it is fixed only up to adding whole multiples of 2π, since $e^{i(\theta+2\pi)} = e^{i\theta}$. Consequently $\log z = \log|z| + i \arg z$ has infinitely many values, and so does $z^w = e^{w \log z}$ when w is not a whole number. A branch of $\log z$ is the single-valued function obtained by confining $\arg z$ to one interval of length 2π, such as $-\pi < \arg z < \pi$. On the ray where $\arg z$ would have to jump from one end of the interval to the other (for the interval $(-\pi, \pi)$, the negative real axis) the branch is discontinuous; that ray is the branch cut, and the branch is analytic on the plane with the cut removed. Crossing the cut changes $\arg z$ by 2π and therefore multiplies z^w by $e^{2\pi i w}$ or $e^{-2\pi i w}$. The whole of Section 11 rests on the two different values of $(-x)^{s-1}$ on the two sides of its cut.

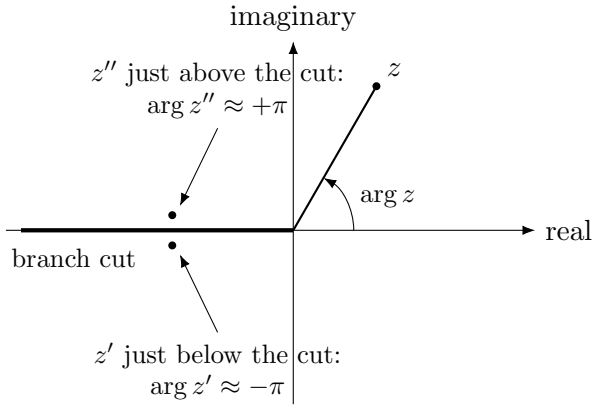


Figure 5: The branch of $\log z = \log|z| + i \arg z$ with $-\pi < \arg z < \pi$. The angle is measured from the positive real axis, counterclockwise positive. The points z'' and z', just above and just below the negative real axis, are close together, but their angles are near $+\pi$ and $-\pi$; so the branch jumps by $2\pi i$ across the negative real axis, which is therefore its branch cut, and the power $z^w = e^{w \log z}$ jumps there by the factor $e^{2\pi i w}$. Section 11 uses the same construction for $\log(-x)$, whose cut is then the positive real x-axis.

4.5 Convergence of series of functions

17. **Uniform convergence; Weierstrass's comparison test.** A series of functions $\sum f_n(s)$ converges uniformly on a set if, for every tolerance ε, there is one stage N beyond which the partial sums are within ε of the sum for every s in the set at once. Weierstrass's test: if there are constants M_n with $|f_n(s)| \le M_n$ for every s in the set and $\sum M_n$ finite, the series converges uniformly there. Consequence (also Weierstrass): a uniformly convergent series of analytic functions is analytic, and may be integrated term by term along any path in the set. Riemann had this from Cauchy's integral theorem; the names are later.

18. **Morera's theorem (1886).** If f is continuous on a region and $\int f(z)\,dz = 0$ around every small triangle in the region, then f is analytic there. It is the converse of Cauchy's theorem, and it is used (Sections 11 and 14) to show that an integral whose integrand depends analytically on a parameter is analytic in that parameter. Morera worked after Riemann; Riemann argued directly from Cauchy's theorem.

4.6 The special functions and numbers

19. **The Gamma function.** $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$ for $\operatorname{Re} s > 0$ (Euler, 1729; the letter is Legendre's, 1811); it satisfies $\Gamma(s+1) = s\Gamma(s)$ and $\Gamma(1) = 1$, so $\Gamma(n+1) = n! = 1\cdot 2\cdots n$. Riemann writes $\Pi(s)$ for $\Gamma(s+1)$, after Gauss (1812). Section 9 proves the properties used.

20. **The Bernoulli numbers.** The coefficients B_k in $\dfrac{x}{e^x-1} = \sum_{k\ge 0}\dfrac{B_k}{k!}x^k$ (Jakob Bernoulli, 1713): $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, and $B_k = 0$ for every odd $k \ge 3$. They are computed in Section 10.

21. **Abel summation.** A rule for attaching a number to a series $\sum a_n$ that diverges: form $\sum a_n x^n$, which may converge for $|x| < 1$, and take the limit of its sum as x increases to 1. Abel's theorem (1826) shows that for a series that converges in the ordinary sense the rule returns the ordinary sum; Euler used the rule in 1749 (Section 7), before Abel was born.

22. **Stirling's formula (1730).** For large $|z|$ away from the negative real axis, $\log\Gamma(z) = (z-\frac{1}{2})\log z - z + \frac{1}{2}\log(2\pi) +$ (a term of size $1/|z|$). It supplies the main term of the zero count in Section 15.

23. **The prime-counting function $\pi(x)$.** The number of primes not exceeding x: $\pi(10) = 4$ (the primes $2, 3, 5, 7$), $\pi(100) = 25$. Here π names a function, not the number $3.14159\ldots$; the notation is Landau's (1909); Riemann writes $F(x)$.

24. **The logarithmic integral $\operatorname{Li}(x)$.** The function $\operatorname{Li}(x) = \int_0^x dt/\log t$, with the integral taken as a principal value across $t = 1$, where $1/\log t$ is infinite (that is, as the limit of $\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x$ as $\varepsilon \to 0$). The letters abbreviate "logarithmic integral" and do not name a person. $\operatorname{Li}(100) \approx 30.1$, $\operatorname{Li}(1000) \approx 177.6$; for large x, $\operatorname{Li}(x)$ is close to $x/\log x$. Gauss found from tables that $\pi(x)$ stays close to $\operatorname{Li}(x)$.

25. **The Möbius function $\mu(n)$ (1832).** $\mu(1) = 1$; $\mu(n) = (-1)^k$ if n is a product of k different primes; $\mu(n) = 0$ if some prime squared divides n. Thus $\mu(2) = \mu(3) = \mu(5) = -1$, $\mu(4) = 0$, $\mu(6) = +1$, $\mu(30) = -1$. Its one essential property, $\sum_{d|n}\mu(d) = 1$ for $n = 1$ and 0 for $n > 1$, and the inversion rule built on it, are proved in Section 16.

26. **Fourier's inversion theorem; the Mellin transform.** For a suitable function f on the real line, the transform $\hat f(\xi) = \int_{-\infty}^\infty f(t)e^{-2\pi i\xi t}dt$ determines f, by the mirror formula $f(t) = \int_{-\infty}^\infty \hat f(\xi)e^{2\pi i\xi t}d\xi$ (Fourier, 1811 and 1822). Substituting $x = e^t$ turns the pair into: $\tilde F(s) = \int_0^\infty F(x)x^{s-1}dx$ determines F by $F(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\tilde F(s)x^{-s}ds$. This second pair is now named after Mellin (1890s); Riemann calls it Fourier's theorem.

4.7 Notation

The exponent is written s; from Section 8 on it is a complex number $s = \sigma + it$ with σ its real part and t its imaginary part, both real. Riemann uses the same letter s and sets $s = \frac{1}{2} + ti$ for the line of his hypothesis. Riemann's factorial function $\Pi(s)$ is $\Gamma(s+1)$, so his $\Pi(s-1)$ is $\Gamma(s)$ and his $\Pi(s/2-1)$ is $\Gamma(s/2)$. Riemann's $\psi(x) = \sum_{n\geq 1} e^{-n^2\pi x}$ is used, together with the symmetric series $\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = 1 + 2\psi(x)$. The letter ζ for the function was introduced by Riemann in this paper.

5 Euler's series and the product over the primes (1737), every step written out

5.1 The series, its terms, and what it means for it to have a value

Throughout this section s is a real number and n is a positive whole number, taking the values $1, 2, 3, \ldots$ in turn. The symbol n^s means $e^{s \log n}$, the number n raised to the real power s; for whole-number s it is the ordinary power, and for other s it is defined by this formula. Its reciprocal is written $n^{-s} = 1/n^s$. Euler's series is the sum of these reciprocals over all n:

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots + \frac{1}{n^s} + \cdots . \tag{1}$$

A series has a value only when its partial sums settle to a limit (item 2 of Section 4). The N-th partial sum of (1) is

$$S_N(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{N^s} = \sum_{n=1}^{N} \frac{1}{n^s}, \tag{2}$$

where N is a positive whole number and the sum has N terms. Every term $1/n^s$ is a positive number, so $S_{N+1}(s) = S_N(s) + 1/(N+1)^s$ is larger than $S_N(s)$: the partial sums increase with N. An increasing sequence has a limit if and only if it is bounded above. So the series (1) has a value exactly when there is a fixed number that every $S_N(s)$ stays below. The next two subsections show that such a number exists when $s > 1$ and does not exist when $s \leq 1$.

5.2 When $s > 1$: the partial sums are bounded, and the bound shows the pole

Fix $s > 1$ and fix a whole number $n \geq 2$. Consider the real variable x on the interval $n - 1 \leq x \leq n$, an interval of length 1. On this interval $x \leq n$. Since $s > 0$, raising to the power s preserves order among positive numbers, so $x^s \leq n^s$. Taking reciprocals reverses order, so

$$\frac{1}{x^s} \geq \frac{1}{n^s} \qquad \text{for every } x \text{ with } n - 1 \leq x \leq n. \tag{3}$$

Integrate both sides of (3) over the interval from $n-1$ to n; the variable of integration is x, and the whole number n enters only as the two endpoints. The integral of the larger function is at least the integral of the smaller. On the right side, $1/n^s$ does not depend on x, so it is a constant on the interval, and the integral of a constant over an interval of length $n - (n-1) = 1$ is that constant:

$$\int_{n-1}^{n} \frac{dx}{x^s} \;\geq\; \int_{n-1}^{n} \frac{dx}{n^s} = \frac{1}{n^s} \int_{n-1}^{n} dx = \frac{1}{n^s} \cdot \big(n - (n-1)\big) = \frac{1}{n^s}. \tag{4}$$

The left side is evaluated by an antiderivative. Since $s \neq 1$, the function $x^{1-s}/(1-s)$ has derivative x^{-s} (the power rule: the derivative of x^a is ax^{a-1}, applied with $a = 1 - s$). So

$$\int_{n-1}^{n} \frac{dx}{x^s} = \left[\frac{x^{1-s}}{1-s}\right]_{x=n-1}^{x=n} = \frac{n^{1-s} - (n-1)^{1-s}}{1-s} = \frac{(n-1)^{1-s} - n^{1-s}}{s-1}, \tag{5}$$

the last equality being multiplication of numerator and denominator by -1. The variable x has gone; what remains is a function of n and s alone, and (4) now reads, with nothing but n and s in it,

$$\frac{1}{n^s} \;\le\; \frac{(n-1)^{1-s} - n^{1-s}}{s-1} \qquad (n \ge 2,\ s > 1). \tag{6}$$

Check with $s = 2$, where $(n-1)^{-1} - n^{-1} = \frac{1}{(n-1)n}$: at $n = 2$ it says $\frac{1}{4} \le \frac{1}{2}$; at $n = 3$, $\frac{1}{9} \le \frac{1}{6}$; at $n = 4$, $\frac{1}{16} \le \frac{1}{12}$. All true, and each right side is the exact area under the curve $y = x^{-2}$ between $n-1$ and n (Figure 6).

Now write (6) for $n = 2$, then for $n = 3$, and so on up to $n = N$, and add the $N-1$ inequalities. On the left the terms $1/2^s, 1/3^s, \ldots, 1/N^s$ add up to $S_N(s) - 1$. On the right the sum telescopes: each n^{1-s} that is subtracted in the n-th term is added back in the $(n+1)$-th term, so

$$\sum_{n=2}^{N} \frac{(n-1)^{1-s} - n^{1-s}}{s-1} = \frac{\left(1^{1-s} - 2^{1-s}\right) + \left(2^{1-s} - 3^{1-s}\right) + \cdots + \left((N-1)^{1-s} - N^{1-s}\right)}{s-1} = \frac{1 - N^{1-s}}{s-1}. \tag{7}$$

The same number is the integral over the joined interval: the integrals over $[1,2], [2,3], \ldots, [N-1, N]$ lie end to end, and their sum is

$$\int_1^N \frac{dx}{x^s} = \left[\frac{x^{1-s}}{1-s}\right]_{x=1}^{x=N} = \frac{N^{1-s} - 1}{1-s} = \frac{1 - N^{1-s}}{s-1}, \tag{8}$$

which is (7) again. Hence

$$S_N(s) - 1 = \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{N^s} \;\le\; \frac{1 - N^{1-s}}{s-1} = \int_1^N \frac{dx}{x^s}. \tag{9}$$

Because $s > 1$, the exponent $1 - s$ is negative, so $N^{1-s} = 1/N^{s-1}$ is a positive number less than 1, and therefore $1 - N^{1-s}$ is a positive number less than 1. Dividing by the positive number $s - 1$,

$$0 < \frac{1 - N^{1-s}}{s-1} < \frac{1}{s-1}. \tag{10}$$

Putting (10) into (9) and adding 1 to both sides,

$$S_N(s) < 1 + \frac{1}{s-1} = \frac{s}{s-1} \qquad \text{for every } N. \tag{11}$$

So the increasing sequence of partial sums stays below the fixed number $s/(s-1)$. It therefore has a limit. That limit is the value of the series, and it is the number Euler worked with and Riemann later named:

$$\zeta(s) = \lim_{N\to\infty} S_N(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s > 1). \tag{12}$$

Since every partial sum is at least its first term 1, and at most $s/(s-1)$, the limit satisfies

$$1 \le \zeta(s) \le \frac{s}{s-1} \qquad (s > 1). \tag{13}$$

The comparison in the other direction. Now take the interval $n \le x \le n+1$ instead, for $n \ge 1$. On it $x \ge n$, so $x^s \ge n^s$, so $1/x^s \le 1/n^s$. Integrate over this interval of length 1, evaluating the left side by the same antiderivative $x^{1-s}/(1-s)$:

$$\frac{n^{1-s} - (n+1)^{1-s}}{s-1} = \int_n^{n+1} \frac{dx}{x^s} \;\le\; \int_n^{n+1} \frac{dx}{n^s} = \frac{1}{n^s} \qquad (n \ge 1,\ s > 1). \tag{14}$$

For $s = 2$ this is $\frac{1}{n(n+1)} \le \frac{1}{n^2}$: at $n = 1$, $\frac{1}{2} \le 1$; at $n = 2$, $\frac{1}{6} \le \frac{1}{4}$. Add (14) for $n = 1, 2, \dots, N$. The right sides add up to $S_N(s)$; the left sides telescope exactly as in (7), the surviving terms being $1^{1-s} = 1$ from $n = 1$ and $-(N+1)^{1-s}$ from $n = N$:

$$\frac{1 - (N+1)^{1-s}}{s-1} = \int_1^{N+1} \frac{dx}{x^s} \;\le\; S_N(s). \tag{15}$$

Let $N \to \infty$: $(N+1)^{1-s} = 1/(N+1)^{s-1}$ tends to 0 because $s - 1 > 0$, so the left side tends to $1/(s-1)$, while the right side tends to $\zeta(s)$. Limits preserve non-strict inequalities, so

$$\frac{1}{s-1} \le \zeta(s) \qquad (s > 1). \tag{16}$$

Combining (16) with the upper bound in (13) and multiplying through by the positive number $s-1$:

$$1 \le (s-1)\,\zeta(s) \le s \qquad (s > 1). \tag{17}$$

What (17) says. Let s decrease towards 1. The lower bound $1/(s-1)$ in (16) grows beyond every number, so $\zeta(s)$ grows beyond every number: the value of the series becomes infinite as $s \to 1$. But the product $(s-1)\zeta(s)$ is squeezed between 1 and s, and $s \to 1$; so $(s-1)\zeta(s) \to 1$. In the vocabulary of Section 4, item 12, a function that behaves near $s = 1$ like a nonzero constant divided by $(s-1)$ has a simple pole at $s = 1$, and the constant is its residue. Equation (17) says that if ζ has a pole at $s = 1$ its residue is exactly 1. Riemann's continuation (Section 12) will show that ζ does have a simple pole there, with residue 1, and no other singularity anywhere; Euler's two comparisons have already fixed the number.

5.3 When $s \le 1$: the partial sums exceed every bound, so the series has no value

Take $s = 1$ first. The partial sums are the harmonic numbers

$$H_N = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}. \tag{18}$$

Repeat the "other direction" comparison with $s = 1$: on $n \le x \le n+1$ one has $1/x \le 1/n$, so $\int_n^{n+1} dx/x \le 1/n$; adding for $n = 1, \dots, N$,

$$\log(N+1) = \int_1^{N+1} \frac{dx}{x} \le H_N, \tag{19}$$

where the integral was evaluated because $\log x$ has derivative $1/x$ and $\log 1 = 0$. The number $\log(N+1)$ exceeds any given bound M as soon as $N + 1 > e^M$. Therefore H_N exceeds every bound: there is no ceiling under which the partial sums stay, so they have no limit, and the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ has no value. That is what "the harmonic series diverges" means, and (19) says how it diverges: slowly, at least as fast as the logarithm. Concretely, H_N first exceeds 2 at $N = 4$, first exceeds 5 at $N = 83$, first exceeds 10 at $N = 12\,367$, and first exceeds 100 at $N \approx 1.5 \times 10^{43}$ (Euler showed in 1734 that $H_N - \log N$ tends to the constant $0.5772\ldots$, so H_N is $\log N$ plus a little more than a half).

Now take any $s < 1$, including negative s. For every $n \ge 1$ one has $n^s \le n^1 = n$ (raising a number at least 1 to a smaller power gives a smaller or equal result), so $1/n^s \ge 1/n$. Adding over $n = 1, \dots, N$: $S_N(s) \ge H_N$. Since H_N exceeds every bound, so does $S_N(s)$. So the series (1) has a value for $s > 1$ and no value for any $s \le 1$; the line $s = 1$ is the boundary of Euler's definition, and for $s \le 1$ the symbol $\zeta(s)$ does not yet mean anything. Giving it a meaning there is Riemann's work.

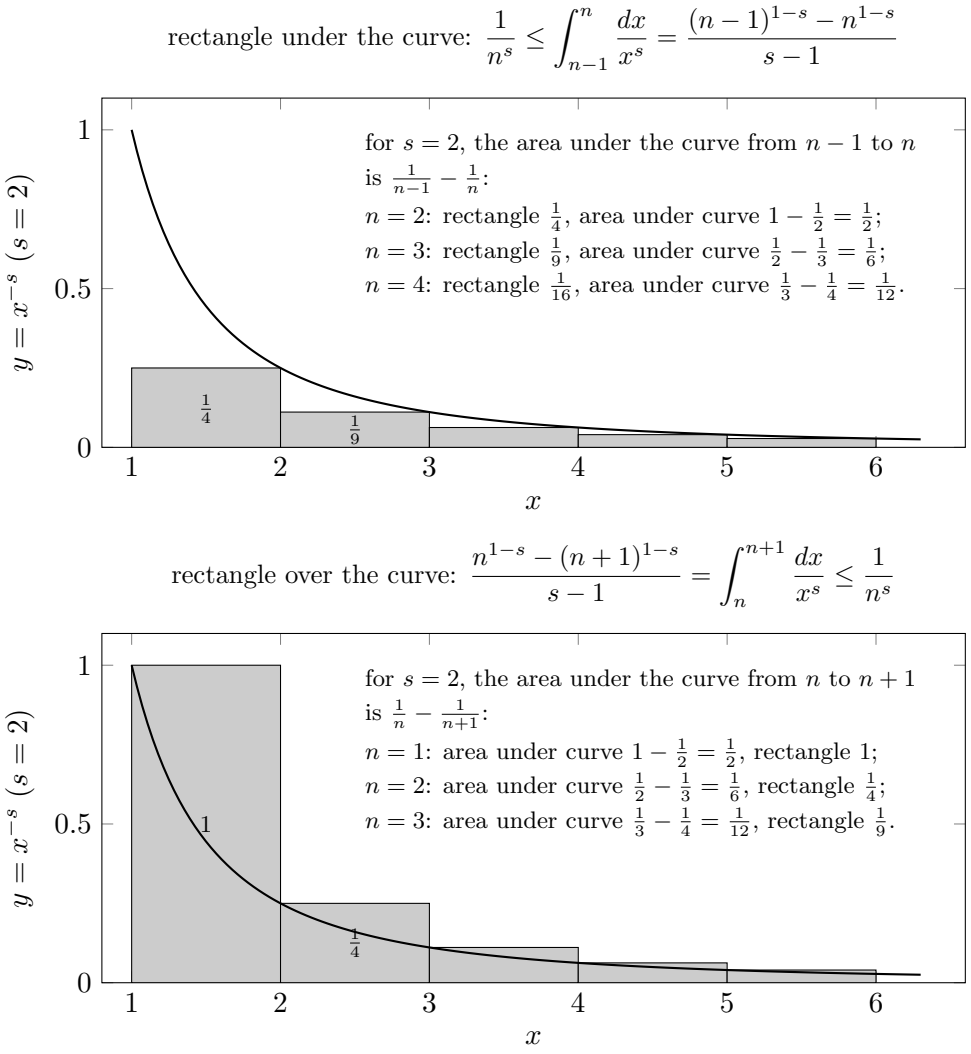


Figure 6: The two comparisons of Section 5.2, drawn for $s = 2$. The curve is $y = x^{-s}$, a function of the real variable x; the whole number n marks where the rectangles stand. Top: on the interval from $x = n - 1$ to $x = n$ the curve stays above the height n^{-s} it has at the right end, so the shaded rectangle of width 1 and height n^{-s} fits under the curve. The area under the curve on that interval is the integral $\int_{n-1}^{n} x^{-s} dx$, whose value, by the antiderivative $x^{1-s}/(1-s)$, is $\big((n-1)^{1-s} - n^{1-s}\big)/(s-1)$; for $s = 2$ this is $\frac{1}{n-1} - \frac{1}{n}$. The rectangle's area n^{-s} is at most that, which is (6). Adding the rectangles for $n = 2, \ldots, N$ gives $S_N(s) - 1$ on one side and the telescoped sum $(1 - N^{1-s})/(s-1)$ on the other, (9). Bottom: on the interval from $x = n$ to $x = n + 1$ the curve stays below the height n^{-s} it has at the left end, so the rectangle covers the curve, and the area under the curve, $\big(n^{1-s} - (n+1)^{1-s}\big)/(s-1)$, is at most n^{-s}, which is (14).

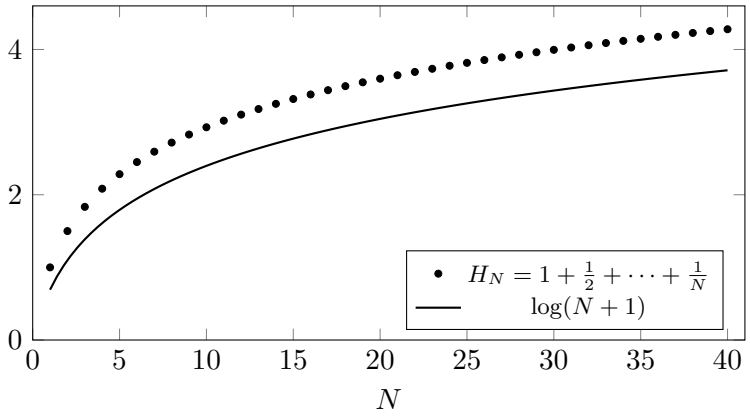


Figure 7: The partial sums H_N of the harmonic series (dots) against the lower bound $\log(N+1)$ of (19) (curve). The dots stay above the curve, and the curve rises without bound, so the dots do too: that is the divergence, and its rate. The gap $H_N - \log N$ settles towards $0.5772\ldots$.

5.4 The product over the primes, proved with finite sums

In this subsection p stands for a prime number, one of $2, 3, 5, 7, 11, 13, \ldots$, and P stands for one particular prime up to which the primes are used. The theorem Euler proved in 1737 is that for $s > 1$

$$\zeta(s) = \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \frac{1}{1-7^{-s}} \cdots = \prod_{p} \frac{1}{1-p^{-s}}, \tag{20}$$

where the product has one factor for every prime, and an infinite product means the limit of the finite products as more and more factors are included. The proof removes from the series, one prime at a time, every term whose index is divisible by that prime, and keeps track of what remains. Every manipulation is done on finite partial sums and then followed by a limit, so no step depends on rearranging an infinite series.

Step 1: removing the even terms. Multiply the partial sum $S_N(s)$ by 2^{-s}. For each term, $2^{-s} \cdot n^{-s} = (2n)^{-s}$, because $2^{-s}n^{-s} = e^{-s\log 2}e^{-s\log n} = e^{-s(\log 2+\log n)} = e^{-s\log(2n)}$. So

$$2^{-s}S_N(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots + \frac{1}{(2N)^s}, \tag{21}$$

which is the sum of m^{-s} over the even numbers m from 2 to $2N$. Subtract this from the partial sum $S_{2N}(s)$, which is the sum of m^{-s} over all m from 1 to $2N$. The even m cancel and the odd m remain:

$$S_{2N}(s) - 2^{-s}S_N(s) = \sum_{\substack{m \le 2N \\ m \text{ odd}}} \frac{1}{m^s} = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots + \frac{1}{(2N-1)^s}. \tag{22}$$

Let $N \to \infty$. On the left, $S_{2N}(s) \to \zeta(s)$ and $S_N(s) \to \zeta(s)$, so the left side tends to $\zeta(s) - 2^{-s}\zeta(s) = (1-2^{-s})\zeta(s)$. The right side is a sum of positive terms whose partial sums are bounded (each is at most $S_{2N}(s) < s/(s-1)$), so it has a limit, which is the sum over all odd m. Hence

$$\Big(1 - \frac{1}{2^s}\Big)\zeta(s) = \sum_{m \text{ odd}} \frac{1}{m^s} = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots. \tag{23}$$

Step 2: the same operation for any prime, in general form. Let T be a set of positive whole numbers with the property that for some prime q, whenever m is in T so is qm, and whenever m is in T and q divides m so is m/q. (The set of all positive whole numbers has this property for every q; the set of odd numbers has it for every odd q.) Write $A_T(s) = \sum_{m \in T} m^{-s}$, a series of

positive terms whose partial sums are bounded by $\zeta(s)$ and which therefore has a value. The claim is

$$\Big(1-\frac{1}{q^s}\Big)A_T(s)=\sum_{\substack{m\in T\\ q\nmid m}}\frac{1}{m^s}, \tag{24}$$

the sum over the members of T not divisible by q. Proof with finite sums: multiply the partial sum of A_T up to N by q^{-s}:

$$q^{-s}\sum_{\substack{m\in T\\ m\le N}}\frac{1}{m^s}=\sum_{\substack{m\in T\\ m\le N}}\frac{1}{(qm)^s}=\sum_{\substack{m'\in T,\ q\mid m'\\ m'\le qN}}\frac{1}{m'^s}.$$

The second equality substitutes $m'=qm$: as m runs over the members of T up to N, the number qm runs over exactly the members of T that are divisible by q and at most qN, because T is closed under multiplying by q and under dividing by q. Now subtract this from the partial sum of A_T up to qN: the members of T divisible by q cancel, the others remain,

$$\sum_{\substack{m\in T\\ m\le qN}}\frac{1}{m^s}-q^{-s}\sum_{\substack{m\in T\\ m\le N}}\frac{1}{m^s}=\sum_{\substack{m\in T,\ q\nmid m\\ m\le qN}}\frac{1}{m^s}.$$

Let $N\to\infty$: the left tends to $A_T(s)-q^{-s}A_T(s)$ and the right to the sum in (24), which exists by the bounded-partial-sums test. This proves (24).

Step 3: using the primes in turn. Apply Step 2 with $q=2$ and T the set of all positive whole numbers: this is Step 1 again, and the result is the sum over the numbers with no factor 2. Apply Step 2 with $q=3$ and T the set of numbers with no factor 2 (closed under multiplying and dividing by 3, since 3 is odd): the result is

$$\Big(1-\frac{1}{3^s}\Big)\Big(1-\frac{1}{2^s}\Big)\zeta(s)=\sum_{\substack{m\ \text{with no factor}\\ 2\ \text{or}\ 3}}\frac{1}{m^s}=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\cdots. \tag{25}$$

Apply Step 2 with $q=5$ to this set, then with $q=7$, and so on through every prime up to P. After the prime P has been used, the set of surviving indices is T_P, the set of positive whole numbers that have no prime factor at most P; it contains 1, and its next members are the primes larger than P. The result is

$$\prod_{p\le P}\Big(1-\frac{1}{p^s}\Big)\cdot\zeta(s)=\sum_{m\in T_P}\frac{1}{m^s}=1+\sum_{\substack{m\in T_P\\ m>1}}\frac{1}{m^s}. \tag{26}$$

all n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
after 2:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
after 3:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
after 5:	**1**	2	3	4	5	6	**7**	8	9	10	**11**	12	**13**	14	15	16	**17**	18	**19**	20	21	22	**23**	24	25	26	27	28	**29**	30

Figure 8: The sieve of Section 5.4, shown on the indices $1,\ldots,30$. Multiplying $\zeta(s)$ by $(1-2^{-s})$ deletes every term whose index is a multiple of 2 (greyed in the second row); multiplying by $(1-3^{-s})$ then deletes the multiples of 3; multiplying by $(1-5^{-s})$ deletes the multiples of 5. After the primes up to $P=5$ the surviving indices (bold) are 1 and the numbers with no prime factor at most 5: the set T_5 of (26). Every survivor other than 1 exceeds P, which is what makes the surviving sum tend to 1.

Step 4: the right side of (26) tends to 1. Every member $m > 1$ of T_P has at least one prime factor, and by the definition of T_P that prime factor is larger than P; a number is at least as large as any of its factors, so $m > P$. Therefore the sum over $m \in T_P$, $m > 1$, is at most the sum of m^{-s} over all $m > P$, and that is bounded by an integral exactly as in (6)–(8), the telescoping now starting at P instead of 1 and running to infinity, so that only the term P^{1-s} survives:

$$0 \le \sum_{\substack{m \in T_P \\ m>1}} \frac{1}{m^s} \le \sum_{m>P} \frac{1}{m^s} \le \int_P^\infty \frac{dx}{x^s} = \frac{P^{1-s}}{s-1}. \tag{27}$$

As P runs through larger and larger primes, $P^{1-s} = 1/P^{s-1}$ tends to 0 because $s - 1 > 0$. So the right side of (26) is squeezed between 1 and $1 + P^{1-s}/(s-1)$, and tends to 1:

$$\lim_{P\to\infty} \prod_{p\le P} \left(1 - \frac{1}{p^s}\right) \cdot \zeta(s) = 1. \tag{28}$$

Step 5: inverting. Each factor $1 - p^{-s}$ lies between $1 - 2^{-s}$ and 1 and is positive, so each finite product is a positive number and may be divided by. Dividing $\zeta(s)$ by the finite product in (28) and letting $P \to \infty$:

$$\lim_{P\to\infty} \prod_{p\le P} \frac{1}{1-p^{-s}} = \lim_{P\to\infty} \frac{\zeta(s)}{\prod_{p\le P}(1-p^{-s})\,\zeta(s)} = \frac{\zeta(s)}{1} = \zeta(s), \tag{29}$$

which is (20), the infinite product being by definition this limit. ■

Two readings of the product. First reading: expand each factor as a geometric series, $1/(1 - p^{-s}) = 1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots$, and multiply the expansions out; the general term of the product is $(p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r})^{-s}$, one for each way of picking a power of each prime. By the unique factorisation of whole numbers into primes, each n arises from exactly one such choice. So the product is a restatement of unique factorisation; this is why the function knows about primes. Second reading, the one used in the rest of this paper: since every partial sum of the series is at least 1, $\zeta(s) \ge 1 > 0$ for every $s > 1$; the function has no zero on the real half-line $s > 1$. Section 8 extends this to the complex half-plane $\operatorname{Re} s > 1$ by the same proof.

5.5 The logarithm of the product, and where the sum of the reciprocals of the primes diverges

Euler's first use of the product was to prove that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \sum_p \frac{1}{p} \tag{30}$$

has partial sums that exceed every bound. The steps are these.

Step 1: the series for $-\log(1-u)$. Let u be a real number with $0 \le u < 1$. The geometric series $1 + u + u^2 + \cdots$ has partial sums $(1 - u^{K+1})/(1-u)$, which tend to $1/(1-u)$ as $K \to \infty$ because $u^{K+1} \to 0$. So $1/(1-v) = \sum_{k\ge0} v^k$ for every v with $0 \le v \le u$, and by Weierstrass's test (item 17 of Section 4, with $M_k = u^k$) the convergence is uniform on $[0, u]$, which permits integrating term by term from 0 to u:

$$-\log(1-u) = \int_0^u \frac{dv}{1-v} = \sum_{k\ge0} \int_0^u v^k\,dv = \sum_{k\ge0} \frac{u^{k+1}}{k+1} = \sum_{k\ge1} \frac{u^k}{k} = u + \frac{u^2}{2} + \frac{u^3}{3} + \cdots, \tag{31}$$

where the first equality holds because $-\log(1-v)$ has derivative $1/(1-v)$ and vanishes at $v = 0$.

Step 2: the logarithm of one factor. For $s > 1$ and any prime p, the number $u = p^{-s}$ satisfies $0 < u \le 2^{-s} < \frac{1}{2}$. Put it into (31):

$$\log \frac{1}{1-p^{-s}} = -\log\left(1-p^{-s}\right) = \sum_{k\ge 1} \frac{p^{-ks}}{k} = \frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \cdots . \tag{32}$$

Step 3: the logarithm of the finite product. The logarithm of a product of positive numbers is the sum of their logarithms, so for the finite product up to P,

$$\log \prod_{p\le P} \frac{1}{1-p^{-s}} = \sum_{p\le P} \sum_{k\ge 1} \frac{p^{-ks}}{k}. \tag{33}$$

As $P \to \infty$ the product tends to $\zeta(s)$ by (29), and $\zeta(s) \ge 1 > 0$; the logarithm is continuous at every positive number, so the left side tends to $\log \zeta(s)$:

$$\log \zeta(s) = \lim_{P\to\infty} \sum_{p\le P} \sum_{k\ge 1} \frac{p^{-ks}}{k}. \tag{34}$$

Step 4: separating the $k = 1$ terms and bounding the rest. Split each inner sum in (33) into its first term and its remainder:

$$\sum_{p\le P} \sum_{k\ge 1} \frac{p^{-ks}}{k} = \sum_{p\le P} \frac{1}{p^s} + \sum_{p\le P} R_p(s), \qquad R_p(s) = \sum_{k\ge 2} \frac{p^{-ks}}{k}. \tag{35}$$

Bound $R_p(s)$: each $1/k$ with $k \ge 2$ is at most 1, so $R_p(s) \le \sum_{k\ge 2} p^{-ks}$; this is a geometric series with first term p^{-2s} and ratio p^{-s}, so it equals $p^{-2s}/(1-p^{-s})$; and since $p^{-s} < \frac{1}{2}$ the denominator exceeds $\frac{1}{2}$, giving $R_p(s) \le 2p^{-2s}$. Add over the primes up to P, enlarge the sum to all whole numbers $m \ge 2$ (more positive terms), and use $p^{-2s} \le p^{-2}$ for $s > 1$:

$$0 \le \sum_{p\le P} R_p(s) \le 2 \sum_{p\le P} \frac{1}{p^{2s}} \le 2 \sum_{m\ge 2} \frac{1}{m^2} \le 2 \int_1^\infty \frac{dx}{x^2} = 2, \tag{36}$$

the last inequality being (9)–(10) with $s = 2$. So the remainder is at most 2, for every P and every $s > 1$.

Step 5: the sum over primes of p^{-s} becomes infinite as s decreases to 1. From (34), (35) and (36), the partial sums $\sum_{p\le P} p^{-s}$ increase with P and are bounded by $\log \zeta(s)$, so they have a limit $\sum_p p^{-s}$, and that limit satisfies

$$\log \zeta(s) - 2 \;\le\; \sum_p \frac{1}{p^s} \;\le\; \log \zeta(s) \qquad (s > 1). \tag{37}$$

Now let s decrease to 1. By (16), $\zeta(s) \ge 1/(s-1)$, so $\log \zeta(s) \ge \log \frac{1}{s-1} = -\log(s-1)$, which grows beyond every bound as $s - 1 \to 0$. By the left inequality in (37), so does $\sum_p p^{-s}$:

$$\sum_p \frac{1}{p^s} \;\ge\; -\log(s-1) - 2 \;\longrightarrow\; \infty \qquad \text{as } s \to 1^+. \tag{38}$$

Step 6: therefore the partial sums of (30) exceed every bound. Suppose, to the contrary, that there were a fixed number B with $\sum_{p\le P} 1/p \le B$ for every P. For $s > 1$ and every prime p one has $p^s > p$, hence $p^{-s} < p^{-1}$; adding over $p \le P$ gives $\sum_{p\le P} p^{-s} \le \sum_{p\le P} 1/p \le B$, and letting $P \to \infty$ gives $\sum_p p^{-s} \le B$ for every $s > 1$. That contradicts (38), which says the left side exceeds B once s is close enough to 1. So no such B exists: the partial sums of $\sum_p 1/p$ exceed every bound, which is what it means for the series to diverge. ■

Where and how fast. The divergence is real but extremely slow. The partial sum $\sum_{p\le P} 1/p$ first exceeds 1 at $P = 5$ (since $\frac{1}{2}+\frac{1}{3}+\frac{1}{5} = 1.033$), first exceeds 2 at $P = 277$, first exceeds 3 at $P = 5\,195\,977$, and first exceeds 4 at $P \approx 1.8 \times 10^{18}$. Mertens proved in 1874 (after Riemann) that $\sum_{p\le P} 1/p = \log\log P + 0.2615\ldots +$ (a term tending to 0); Euler had written the sum as "$\log\log\infty$" in 1737. The quantity that diverges is the partial sum as a function of P; the place where it diverges is $P \to \infty$; the rate is the logarithm of the logarithm, the slowest divergence in this paper.

What this has to do with Riemann. The argument just given extracted a fact about the primes (there are enough of them that the sum of their reciprocals is infinite) from one fact about the function ζ: that it becomes infinite as s decreases to 1, in the manner fixed by (17). The behaviour of ζ at the edge $s = 1$ of the region where Euler's series defines it is thus a statement about the primes. A finer statement about the primes, the actual count of primes below a given bound, requires knowing the function on the other side of that edge, where the series gives nothing. Riemann's paper is the construction of the function there.

5.6 Euler's values at the even whole numbers

Euler had also found, in 1735 for $s = 2$ and then in general, that

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(6) = \frac{\pi^6}{945}, \qquad \text{and in general} \qquad \zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2\,(2k)!} \tag{39}$$

for every positive whole number k, where B_{2k} is the Bernoulli number of index $2k$ (item 20 of Section 4; for $k = 2$ the formula gives $-\frac{16\pi^4\cdot(-1/30)}{2\cdot 24} = \frac{\pi^4}{90}$). Euler obtained these by writing $\sin x/x$ as a product over its zeros $\pm\pi, \pm 2\pi, \ldots$ and comparing the coefficients of the expanded product with the power series of $\sin x/x$. These values are not needed to construct the continuation, but they are the data against which Euler tested his 1749 relation (next section), and the continuation must reproduce them; Section 12 checks that it does.

6 Where the primes enter the calculus, and where they come out

A reader who sees integrals over a continuous variable on one page and the list $2, 3, 5, 7, 11, \ldots$ on the next is entitled to ask what the one has to do with the other, and in particular what device restricts a sum to prime indices. The answer has four parts; each is a specific equation, already written or to be written below, and Figure 10 draws the whole route.

6.1 What a sum over primes is

A prime is a whole number greater than 1 whose only positive divisors are 1 and itself: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and so on. No formula is used in this paper to produce the primes, and none is needed. Functions that select the primes do exist, and Section 17 exhibits three of them, including one built from ζ itself, and proves what cannot exist; none of them is evaluated anywhere in Euler's or Riemann's argument. The symbol $\sum_p$ means: take the list of primes, in increasing order, and add one term for each entry; $\prod_p$ means multiply one factor for each entry; $\sum_{p\le P}$ means stop at the prime P. The restriction to primes is a restriction of the index set, not the effect of a function. Three facts about the list are used.

The list is infinite (Euclid, *Elements* IX.20, about 300 BC). Given any finite list of primes $p_1, \ldots, p_k$, the number $M = p_1 p_2 \cdots p_k + 1$ leaves remainder 1 on division by each p_j, so none of them divides M; but $M > 1$ has some prime divisor (next fact); that prime is not on the list. So no finite list is complete.

Every whole number $n > 1$ has a prime divisor. Among the divisors of n that exceed 1 there is a smallest, d. If d had a divisor e with $1 < e < d$, then e would also divide n and be a smaller divisor exceeding 1, contradicting the choice of d. So d is prime. This is the only fact about primes

that the proof of the product formula in Section 5.4 used: at Step 4, a surviving index $m > 1$ had a prime divisor, and that divisor exceeded P.

Unique factorisation (Euclid VII and IX; stated and proved in full generality by Gauss, *Disquisitiones Arithmeticae*, 1801, article 16). Every $n > 1$ is a product of primes, and the list of primes in the product, with repetitions, is the same whichever way the factoring is carried out. This fact is not needed for the proof of the product formula; it is what the product formula *says*, as the first reading at the end of Section 5.4 explained: expanding each factor $1/(1-p^{-s})$ as $1+p^{-s}+p^{-2s}+\cdots$ and multiplying out produces each n^{-s} exactly once because each n has exactly one factorisation.

6.2 The one door through which the primes enter

The series $\zeta(s) = \sum_{n\geq 1} n^{-s}$ has one term for every whole number n; nothing in it mentions primes. The product $\prod_p (1-p^{-s})^{-1}$ has one factor for every prime; nothing in it mentions composite numbers. The sieve of Section 5.4 (Figure 8) shows that the two are the same number: multiplying the series by $(1-p^{-s})$ deletes the terms whose index is a multiple of p, and deleting the multiples of each prime in turn deletes every index except 1. The equality

$$\sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

is therefore the statement that the primes generate all whole numbers under multiplication, written as an equation between two functions of s. It is the only place in Sections 5 to 15 where the word "prime" does any work, and every later statement about primes passes back through it.

6.3 What the calculus is for

The calculus never touches an individual prime. It acts on functions of a continuous variable: it differentiates them, integrates them, and takes their limits. The series $\sum n^{-s}$ is such a function of the continuous variable s, and so is the product. The product identity is a dictionary: a fact proved about the function on the series side, by calculus, becomes, when read on the product side, a fact about the primes. Section 5.5 is the first instance, and it is worth seeing its two halves separately. The calculus half: comparing the series with the integral $\int_1^\infty x^{-s}dx$ (Figure 6) shows that $\zeta(s) \geq 1/(s-1)$, which becomes infinite as s decreases to 1. The arithmetic half: the logarithm of the product is $\sum_p p^{-s}$ plus a remainder below 2, so the same quantity becoming infinite forces $\sum_p p^{-s}$ to become infinite, and that forces the partial sums of $\sum_p 1/p$ to exceed every bound. No prime was differentiated or integrated; a function was, and the dictionary translated the result.

6.4 The second door, where the primes come back out

To get the count of primes below x, rather than the divergence of a sum over all primes, Riemann needs the dictionary in the other direction, with more detail. The logarithm of the product (Section 5.5, Step 2, and Section 16 for complex s) is

$$\log \zeta(s) = \sum_p \sum_{n\geq 1} \frac{1}{n} p^{-ns},$$

a series whose terms are indexed by the prime powers p^n: the numbers 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, and so on, each weighted by $1/n$ where n is its exponent. To bring the calculus to bear on this sum over a discrete set, the set is encoded in a function of a continuous variable: the step function

$$J(x) = \sum_{p^n \leq x} \frac{1}{n},$$

which is 0 for $x < 2$, jumps by 1 at every prime, by $\frac{1}{2}$ at every prime square, by $\frac{1}{3}$ at every prime cube, and is constant in between (Figure 9). The jumps of J are the prime powers; the heights of

the jumps are the weights. Section 16 proves, step by step, that

$$\log \zeta(s) = s \int_1^\infty J(x)\, x^{-s-1}\, dx \qquad (\operatorname{Re} s > 1),$$

an integral over the continuous variable x in which the primes are present only as the places where J jumps, and then proves the inverse statement, that $J(x)$ is recovered from $\log \zeta$ by an integral along a vertical line in the complex s-plane. Evaluating that inverse integral is where the analytic continuation earns its keep: the line can be moved across the plane only if ζ, and so $\log \zeta$, is known across the plane, and the obstacles to moving it are exactly the points where $\log \zeta$ is singular, namely the pole at $s = 1$ and the zeros of ζ. Each obstacle leaves a term behind, and those terms are the explicit formula.

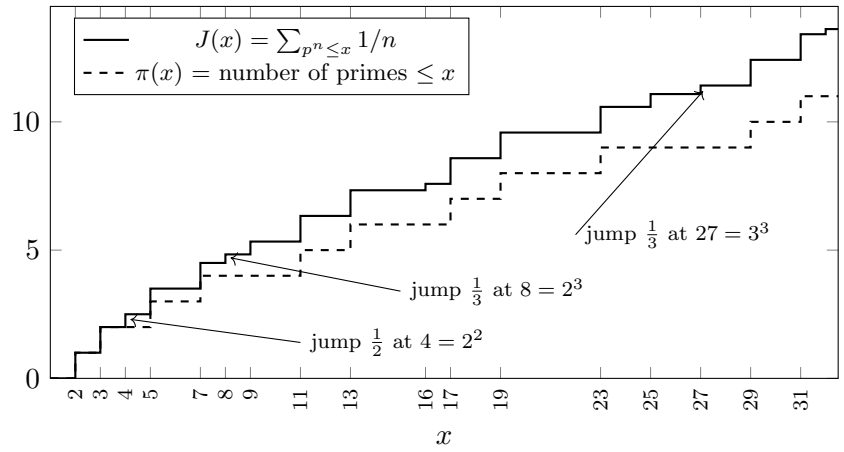


Figure 9: The step function $J(x)$ (solid) and the prime-counting function $\pi(x)$ (dashed) for $1 \le x \le 32$. Both are 0 for $x < 2$ and constant between jumps. $\pi(x)$ jumps by 1 at each prime $2, 3, 5, 7, 11, \ldots$. $J(x)$ jumps by 1 at each prime, by $\frac{1}{2}$ at each prime square $4, 9, 25$, by $\frac{1}{3}$ at each prime cube $8, 27$, by $\frac{1}{4}$ at 16 and by $\frac{1}{5}$ at 32; so $J(10) = 4 + \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 1 = 5\frac{1}{3}$. The places where J jumps are the prime powers: this is how a discrete set is handed to the calculus as a function of the continuous variable x.

6.5 Summary

No function is applied to the index to make it prime; the primes are a set, the whole numbers are generated from that set by multiplication; and the sieve identity is that fact written as an equality of two functions of s. Everything the calculus does afterwards it does to functions of s or of x: the series, the product, the loop integral, the step function. The identity translates each result about the functions into a result about the primes, and the step function J carries the primes, as the positions of its jumps, into the integrals of Section 16.

7 Euler's 1749 relation between the series at s and at $1 - s$, and why it was not yet a theorem

Euler guessed the functional equation of ζ one hundred and ten years before Riemann proved it, and the form of his guess shows exactly what was missing. In this section s is again real, n runs over $1, 2, 3, \ldots$, and x is a real number with $0 \le x < 1$.

7.1 The alternating series and its relation to ζ

Euler worked with the alternating series

$$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{40}$$

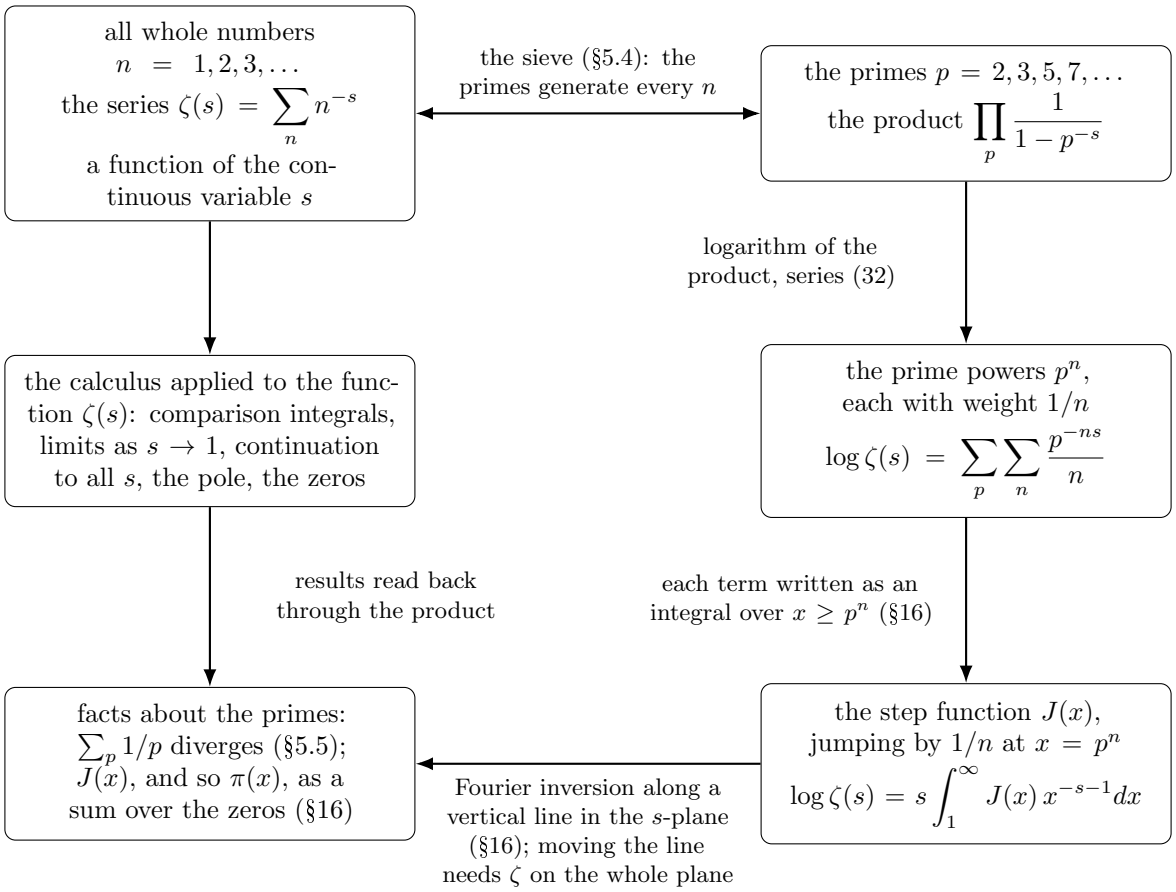


Figure 10: The route from the primes into the calculus and back. The primes enter at one door only, the sieve identity between the series over all n and the product over the primes p. The calculus works on functions of the continuous variables s and x. The primes come out through the step function $J(x)$, whose jumps sit at the prime powers, and through the inversion of the integral that encodes it.

whose terms alternate in sign. For $s > 0$ the terms decrease in size to 0, and a series of that kind converges (its partial sums after an odd and after an even number of terms close in on each other from above and below). For $s > 1$ the relation to ζ is obtained by the operation of Step 1 of Section 5.4: the sum of ζ's terms over even n is $2^{-s}\zeta(s)$, and η is ζ with the even terms counted negatively instead of positively, that is, with twice the even terms subtracted:

$$\eta(s) = \zeta(s) - 2\cdot 2^{-s}\zeta(s) = \big(1 - 2^{1-s}\big)\zeta(s) \qquad (s > 1). \tag{41}$$

(With finite sums: $\sum_{n\le 2N}(-1)^{n-1}n^{-s} = S_{2N}(s) - 2\cdot 2^{-s}S_N(s)$, and $N \to \infty$.)

7.2 Attaching values to the series for $s \le 0$

For $s \le 0$ the terms of (40) do not shrink, so the series does not converge: for $s = 0$ it is $1 - 1 + 1 - 1 + \cdots$, whose partial sums alternate between 1 and 0; for $s = -1$ it is $1 - 2 + 3 - 4 + \cdots$, whose partial sums are $1, -1, 2, -2, 3, \ldots$. Euler attached a value to each by the rule now called Abel summation (item 21 of Section 4): insert a factor x^n into the n-th term, sum the resulting power series in closed form for $0 \le x < 1$, and set $x = 1$ in the closed form. Write $s = -m$ with m a non-negative whole number, so that the n-th term becomes $(-1)^{n-1}n^m x^n$.

$m = 0$. The series $x - x^2 + x^3 - \cdots$ is geometric with first term x and ratio $-x$, so for $0 \le x < 1$

$$f_0(x) = \sum_{n\ge 1}(-1)^{n-1}x^n = \frac{x}{1+x}. \tag{42}$$

At $x = 1$ the closed form gives $\frac{1}{2}$. So Euler's value is $\eta(0) = \frac{1}{2}$.

$m = 1$. Differentiate (42) term by term (permitted inside the interval of convergence) and multiply by x; the term $(-1)^{n-1}x^n$ becomes $(-1)^{n-1}n\,x^n$. On the closed form, the quotient rule gives $\frac{d}{dx}\frac{x}{1+x} = \frac{(1+x)\cdot 1 - x\cdot 1}{(1+x)^2} = \frac{1}{(1+x)^2}$. Hence

$$f_1(x) = \sum_{n\ge 1}(-1)^{n-1}n\,x^n = x\cdot\frac{1}{(1+x)^2} = \frac{x}{(1+x)^2}, \tag{43}$$

and at $x = 1$ this is $\frac{1}{4}$. Euler's value: $\eta(-1) = \frac{1}{4}$.

$m = 2$. Repeat: $\frac{d}{dx}\frac{x}{(1+x)^2} = \frac{(1+x)^2 - x\cdot 2(1+x)}{(1+x)^4} = \frac{(1+x)-2x}{(1+x)^3} = \frac{1-x}{(1+x)^3}$, so

$$f_2(x) = \sum_{n\ge 1}(-1)^{n-1}n^2x^n = \frac{x(1-x)}{(1+x)^3}, \tag{44}$$

which is 0 at $x = 1$. Euler's value: $\eta(-2) = 0$.

$m = 3$. Repeat once more: the numerator of the derivative of (44) is

$$(1-2x)(1+x)^3 - x(1-x)\cdot 3(1+x)^2;$$

dividing numerator and denominator by $(1+x)^2$ leaves $\frac{(1-2x)(1+x)-3x(1-x)}{(1+x)^4} = \frac{1+x-2x-2x^2-3x+3x^2}{(1+x)^4} = \frac{1-4x+x^2}{(1+x)^4}$, so

$$f_3(x) = \sum_{n\ge 1}(-1)^{n-1}n^3x^n = \frac{x(1-4x+x^2)}{(1+x)^4}, \tag{45}$$

which at $x = 1$ is $\frac{1-4+1}{16} = -\frac{2}{16} = -\frac{1}{8}$. Euler's value: $\eta(-3) = -\frac{1}{8}$.

The rule cannot be applied to the series $\sum n^m$ itself (that is, to ζ at $-m$) because $\sum n^m x^n$ tends to $+\infty$ as $x \to 1$; this is why Euler's whole computation runs through the alternating series and reaches ζ only through (41).

7.3 Euler's relation and its check at the whole numbers

For positive even arguments the values were known from Section 5.6 and (41): $\eta(2) = (1-2^{-1})\zeta(2) = \frac{1}{2}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{12}$ and $\eta(4) = (1-2^{-3})\zeta(4) = \frac{7}{8}\cdot\frac{\pi^4}{90} = \frac{7\pi^4}{720}$. Euler compared the attached values at $1-n$ with the known values at n and found that the ratios fit one expression for every whole number n he tried:

$$\frac{\eta(1-n)}{\eta(n)} = -\frac{(n-1)!\,(2^n-1)}{(2^{n-1}-1)\,\pi^n}\cos\frac{n\pi}{2}. \tag{46}$$

Check for $n=2$: left side $\frac{1/4}{\pi^2/12} = \frac{12}{4\pi^2} = \frac{3}{\pi^2}$; right side $-\frac{1!\cdot(4-1)}{(2-1)\pi^2}\cos\pi = -\frac{3}{\pi^2}\cdot(-1) = \frac{3}{\pi^2}$. Check for $n=4$: left side $\frac{-1/8}{7\pi^4/720} = -\frac{720}{56\pi^4} = -\frac{90}{7\pi^4}$; right side $-\frac{3!\cdot(16-1)}{(8-1)\pi^4}\cos 2\pi = -\frac{6\cdot 15}{7\pi^4} = -\frac{90}{7\pi^4}$. For odd $n \geq 3$ both sides are 0: $\cos(n\pi/2) = 0$, and the attached values $\eta(-2), \eta(-4), \ldots$ are 0 (the pattern of (44) continues). Euler then conjectured that (46) holds with the whole number n replaced by an arbitrary real s and the factorial $(n-1)!$ replaced by the function $\Gamma(s)$ that he had himself constructed to take the value $(s-1)!$ at whole numbers (Section 9).

7.4 This is the functional equation

Substitute (41) into (46) with n replaced by s, using it once at s and once at $1-s$: $\eta(s) = (1-2^{1-s})\zeta(s)$ and $\eta(1-s) = (1-2^{1-(1-s)})\zeta(1-s) = (1-2^s)\zeta(1-s)$. The relation becomes

$$\frac{(1-2^s)\,\zeta(1-s)}{(1-2^{1-s})\,\zeta(s)} = -\frac{\Gamma(s)\,(2^s-1)}{(2^{s-1}-1)\,\pi^s}\cos\frac{\pi s}{2}. \tag{47}$$

Multiply both sides by $(1-2^{1-s})\zeta(s)$:

$$(1-2^s)\,\zeta(1-s) = -\frac{\Gamma(s)\,(2^s-1)\,(1-2^{1-s})}{(2^{s-1}-1)\,\pi^s}\cos\frac{\pi s}{2}\;\zeta(s). \tag{48}$$

Two rewritings of the powers of 2: on the left, $1-2^s = -(2^s-1)$; on the right, $1-2^{1-s} = 1-2\cdot 2^{-s} = 2^{-s}(2^s-2) = 2^{-s}\cdot 2\,(2^{s-1}-1) = 2^{1-s}(2^{s-1}-1)$. Put both in:

$$-(2^s-1)\,\zeta(1-s) = -\frac{\Gamma(s)\,(2^s-1)\,2^{1-s}(2^{s-1}-1)}{(2^{s-1}-1)\,\pi^s}\cos\frac{\pi s}{2}\;\zeta(s). \tag{49}$$

Cancel the factor $-(2^s-1)$ from both sides (it is nonzero for $s \neq 0$) and the factor $2^{s-1}-1$ from numerator and denominator on the right (nonzero for $s \neq 1$):

$$\zeta(1-s) = \frac{2^{1-s}}{\pi^s}\cos\frac{\pi s}{2}\,\Gamma(s)\,\zeta(s) = 2\,(2\pi)^{-s}\cos\frac{\pi s}{2}\,\Gamma(s)\,\zeta(s), \tag{50}$$

the last step writing $2^{1-s}\pi^{-s} = 2\cdot 2^{-s}\pi^{-s} = 2(2\pi)^{-s}$. This is, letter for letter, the equation Riemann proves in Section 13 by moving a contour. So the content of the functional equation was in Euler's hands in 1749.

7.5 Why it was not yet a theorem

For $s > 1$ the right side of (50) is a convergent series multiplied by known functions. The left side, $\zeta(1-s)$, is the series $\sum n^{-(1-s)} = \sum n^{s-1}$, whose terms grow without bound (Section 5.3); it has no value, and the number Euler wrote for it is the output of the summation rule applied to the alternating series and then converted through (41). The relation therefore asserted an equality between a function and a convention. For non-whole-number s neither side had any meaning outside the strip $0 < s < 1$, where both alternating series converge but neither could be evaluated in closed form. The equation could become a theorem only after someone supplied a single definition of $\zeta(s)$ that (i) is a convergent, computable expression at every s, (ii) agrees with the series $\sum n^{-s}$ wherever that series converges, and (iii) agrees with Euler's attached values at the negative whole numbers as a consequence rather than as a stipulation. Riemann's paper supplies that definition and derives (ii) and (iii) from it. That is what "finding the analytic continuation" means in practice. Riemann cites Euler only for the product, not for the 1749 relation; whether he knew the 1749 paper is not settled by the documents, and nothing in his derivation depends on it.

8 What Riemann brought: the complex exponent, and three facts about functions of a complex variable

Riemann's first move, made in the opening sentence of the 1859 paper, is to let the exponent s be a complex number. From here on $s = \sigma + it$ with σ and t real; $\sigma = \operatorname{Re} s$ is the real part and $t = \operatorname{Im} s$ the imaginary part. The number n^{-s} for a positive whole number n is defined, as before, by

$$n^{-s} = e^{-s \log n}, \tag{51}$$

with $\log n$ the ordinary real logarithm; since e^w is defined for every complex w by its power series, this makes sense for every complex s. Everything else in the paper follows from three facts about functions of a complex variable that Riemann had established in his 1851 dissertation and here uses without comment. They are stated and proved in the vocabulary of Section 4.

8.1 Fact 1: the series is analytic on the half-plane $\operatorname{Re} s > 1$, the product holds there, and the function has no zero there

The modulus of a term. Write $-s \log n = -\sigma \log n - it \log n$ and use $e^{a+b} = e^a e^b$:

$$n^{-s} = e^{-\sigma \log n} \cdot e^{-it \log n} = n^{-\sigma} \cdot e^{-it \log n}.$$

For real θ, $e^{i\theta} = \cos\theta + i \sin\theta$ has modulus $\sqrt{\cos^2\theta + \sin^2\theta} = 1$. So the modulus of the second factor is 1, and

$$\left| n^{-s} \right| = n^{-\sigma} = \frac{1}{n^{\operatorname{Re} s}}. \tag{52}$$

The size of a term depends on the real part of s alone.

Absolute and uniform convergence. By (52) the series of moduli $\sum |n^{-s}|$ is the real series $\sum n^{-\sigma}$, which converges when $\sigma > 1$ by Section 5.2. So $\sum n^{-s}$ converges absolutely for every s with $\operatorname{Re} s > 1$, and its sum is denoted $\zeta(s)$; for real s it is the number of (12). Now fix $\delta > 0$ and consider all s with $\sigma \geq 1 + \delta$. For such s, $|n^{-s}| = n^{-\sigma} \leq n^{-1-\delta}$, a bound independent of s, and $\sum n^{-1-\delta}$ is finite. By Weierstrass's test (item 17 of Section 4) the series converges uniformly on the set $\sigma \geq 1 + \delta$.

Each term is analytic; so is the sum. The function $s \mapsto e^{-s \log n}$ has the complex derivative $-\log n \cdot e^{-s \log n}$ at every s (the chain rule applied to e^w, whose derivative is e^w), so each term is an entire function. The sum is analytic on the open half-plane $\operatorname{Re} s > 1$ by the following argument, which is the one used at every later point where an infinite sum or an integral is shown to be analytic. Take any point s_0 with $\operatorname{Re} s_0 > 1$ and a closed disc around it inside the half-plane; on that disc the convergence is uniform (choose δ smaller than the distance from the disc to the line $\sigma = 1$). Uniform convergence of continuous functions gives a continuous sum. For any small triangle Δ inside the disc, uniform convergence permits integrating term by term, so $\int_\Delta \zeta(s)\, ds = \sum_n \int_\Delta n^{-s}\, ds$; each term on the right is 0 by Cauchy's theorem (item 10), since n^{-s} is entire; hence $\int_\Delta \zeta(s)\, ds = 0$ for every small triangle; by Morera's theorem (item 18) ζ is analytic on the disc, and in particular at s_0. Since s_0 was arbitrary, ζ is analytic on $\operatorname{Re} s > 1$.

The product. Every step of the proof in Section 5.4 was a manipulation of finite sums followed by a limit, and the limits used two things: that the partial sums $S_N(s)$ converge (now true by absolute convergence), and that the tail $\sum_{m>P} |m^{-s}| = \sum_{m>P} m^{-\sigma}$ is at most $P^{1-\sigma}/(\sigma - 1)$ and tends to 0 (true for $\sigma > 1$ by (27)). The identity (24) holds word for word with complex s, since it used only $q^{-s} m^{-s} = (qm)^{-s}$ and convergence. So

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \qquad (\operatorname{Re} s > 1), \tag{53}$$

and in particular the finite-product identity (26) holds for complex s.

No zero in $\operatorname{Re} s > 1$**.** From (26) with complex s and the tail bound,

$$\Big|\prod_{p\le P}\big(1-p^{-s}\big)\cdot\zeta(s)-1\Big| = \Big|\sum_{\substack{m\in T_P\\ m>1}} m^{-s}\Big| \le \sum_{m>P} m^{-\sigma} \le \frac{P^{1-\sigma}}{\sigma-1}.$$

Choose P so large that the right side is less than 1; then the number $\prod_{p\le P}(1-p^{-s})\zeta(s)$ lies within distance less than 1 of 1 and so is not 0. A product of finitely many numbers is nonzero only if each factor is nonzero, so $\zeta(s)\neq 0$. Hence

$$\zeta(s)\neq 0 \qquad \text{for every } s \text{ with } \operatorname{Re} s > 1. \tag{54}$$

The consequence for everything that follows: whatever zeros the continued function has, they lie in the half-plane $\operatorname{Re} s \le 1$.

8.2 Fact 2: an analytic continuation, if it exists, is unique

A point a is a limit point of a set E if every disc around a, however small, contains a point of E other than a. (Every point of a region is a limit point of the region; the point 1 is a limit point of the set $\{1+\frac{1}{n}\}$.)

Theorem 8.1. *Let D be a region, let f and g be analytic on D, and suppose $f(z)=g(z)$ for every z in a set $E\subset D$ that has a limit point a in D. Then $f(z)=g(z)$ for every z in D.*

Proof. Put $h=f-g$, analytic on D, and let Z be the set of points z of D at which $h(z)=0$, $h'(z)=0$, $h''(z)=0$, and so on for every derivative. Three things are shown about Z.

Z is closed in D: each condition $h^{(k)}(z)=0$ defines a closed set (the derivative is continuous), and an intersection of closed sets is closed.

Z is open: at a point z_0 of Z the power series of h around z_0 (item 8 of Section 4) has coefficients $h^{(k)}(z_0)/k!$, all 0, so h is identically 0 on a disc around z_0, and so are all its derivatives; that disc lies in Z.

Z is not empty; indeed $a\in Z$. Suppose not: then some derivative of h is nonzero at a; let m be the smallest order with $h^{(m)}(a)\neq 0$. The power series of h at a then starts with the term of order m, so $h(z)=(z-a)^m u(z)$ with u analytic and $u(a)=h^{(m)}(a)/m!\neq 0$. By continuity u is nonzero on some disc around a, so on that disc h vanishes only at a itself. But a is a limit point of E, and h vanishes on E, so every disc around a contains zeros of h other than a: a contradiction. Hence $a\in Z$.

A region is connected, and a subset of a connected set that is nonempty, open and closed is the whole set. So $Z=D$, which means $h=0$ on D, that is, $f=g$ on D. □

Consequence. Suppose two formulas each define an analytic function on a region containing the half-plane $\operatorname{Re} s>1$, and each agrees with $\sum n^{-s}$ on that half-plane. The half-plane is a set with limit points in the region, so by the theorem the two formulas define the same function on the whole region. This is why Riemann can give two different constructions (the loop integral of Section 11 and the theta integral of Section 14) and call both of them $\zeta(s)$: they are forced to coincide wherever both are defined. It is also why an identity between analytic functions, once proved on an open set, holds wherever both sides are analytic: the functional equation will be proved for $\operatorname{Re} s<0$ and then holds everywhere by the theorem.

8.3 Fact 3: Cauchy's theorem and the residue theorem

The integral of an analytic function around a closed path inside its region is 0 (item 10 of Section 4). Hence a path of integration may be moved through a region where the integrand is analytic without changing the value of the integral, and the integral around a path that encloses finitely many isolated poles equals $2\pi i$ times the sum of the residues at those poles when the path is run counterclockwise (item 14). Riemann uses the moving of a path twice: inward, to collapse a loop onto the positive real axis (Section 11), and outward, to expand it onto the poles of $1/(e^x-1)$ (Section 13).

8.4 The strategy that the three facts make possible

Suppose one can write, for $\operatorname{Re} s > 1$,

$$\zeta(s) = \frac{F(s)}{G(s)}, \tag{55}$$

where F is an entire function and G is a known meromorphic function. Then F/G is meromorphic on the whole plane, agrees with the series on $\operatorname{Re} s > 1$, and is therefore, by Fact 2, the one and only continuation of ζ to the plane minus the points where G vanishes without F also vanishing. The whole problem of continuation is reduced to producing such an F. Riemann produces it as an integral in which the dependence on s sits entirely in a factor $(-x)^{s-1}$ evaluated at points x kept away from 0, on a path along which the rest of the integrand decays exponentially; such an integral is entire in s by the Morera argument used above for the series. The next three sections carry this out.

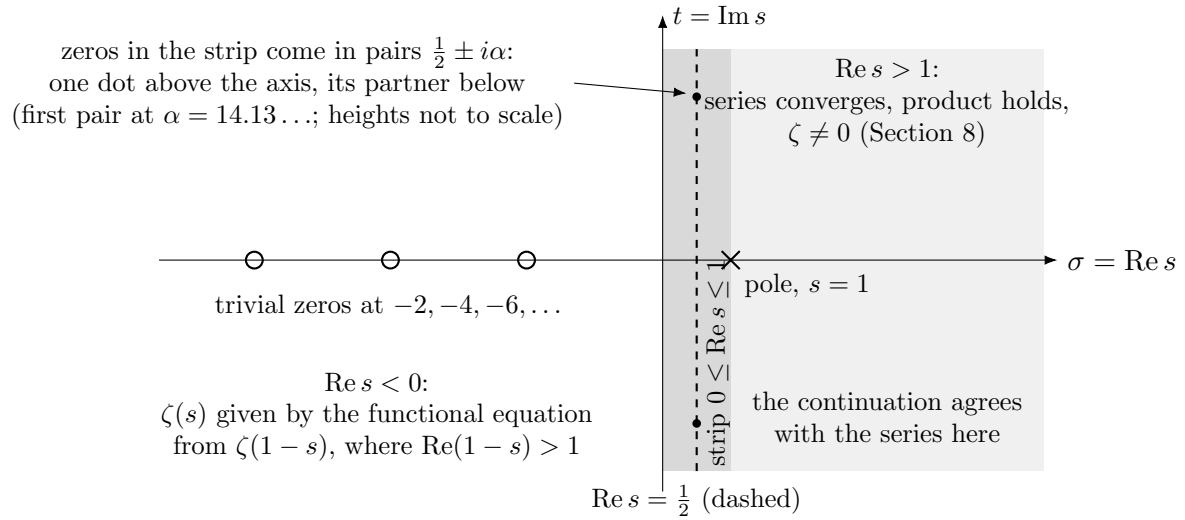


Figure 11: The map of the s-plane assembled in Sections 8 to 13. Light shading: the half-plane $\operatorname{Re} s > 1$ where Euler's series converges and the product shows $\zeta \neq 0$. Darker shading: the strip $0 \leq \operatorname{Re} s \leq 1$, the only place where zeros other than the trivial ones can lie; the dashed line $\operatorname{Re} s = \frac{1}{2}$ is its axis of symmetry and the line of Riemann's hypothesis. The cross at $s = 1$ is the single pole. The circles at $-2, -4, -6, \ldots$ are the trivial zeros. On $\operatorname{Re} s < 0$ the function is determined by its values on $\operatorname{Re} s > 1$ through the functional equation.

9 The Gamma function, and the two identities of Π that Riemann calls known

Riemann's first operation on the series uses Euler's Gamma function, and his symmetric form of the functional equation uses two identities of it. Both are proved here from the integral, by the same tools (a substitution, Cauchy's theorem, Fact 2) that the rest of the paper uses.

9.1 Definition and convergence

For a complex number s with $\operatorname{Re} s > 0$ define

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx, \tag{56}$$

where x is a real variable of integration running from 0 to ∞ and $x^{s-1} = e^{(s-1)\log x}$ with $\log x$ real. The integrand has modulus $e^{-x}x^{\sigma-1}$ by the computation (52) applied to x in place of n. At the end $x \to \infty$: $e^{-x}x^{\sigma-1} \leq e^{-x/2}$ once x is large (because $x^{\sigma-1}e^{-x/2} \to 0$), and $\int^\infty e^{-x/2}dx$ is finite, so the

integral converges there for every s. At the end $x \to 0$: e^{-x} is between e^{-1} and 1, so the integrand is between $e^{-1}x^{\sigma-1}$ and $x^{\sigma-1}$, and $\int_0^1 x^{\sigma-1}dx$ is $1/\sigma$ when $\sigma > 0$ and infinite when $\sigma \le 0$ (item 5 of Section 4). So the integral (56) converges exactly when $\operatorname{Re} s > 0$. This is Riemann's $\Pi(s-1)$, Gauss's notation.

9.2 The recursion, and the values at whole numbers

Integrate (56) with $s+1$ in place of s by parts, differentiating x^s and integrating e^{-x}:

$$\Gamma(s+1) = \int_0^\infty e^{-x}x^s\,dx = \Big[-e^{-x}x^s\Big]_{x=0}^{x=\infty} + s\int_0^\infty e^{-x}x^{s-1}\,dx = 0 + s\,\Gamma(s). \tag{57}$$

The boundary term vanishes at $x = \infty$ because e^{-x} beats any power, and at $x = 0$ because $|x^s| = x^\sigma \to 0$ for $\sigma > 0$. Also $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$. Applying (57) repeatedly: $\Gamma(2) = 1\cdot\Gamma(1) = 1$, $\Gamma(3) = 2\cdot\Gamma(2) = 2$, $\Gamma(4) = 3\cdot\Gamma(3) = 6$, and in general $\Gamma(n+1) = n\cdot(n-1)\cdots 2\cdot 1 = n!$. This is the sense in which Γ interpolates the factorial, and the reason Riemann's $\Pi(n) = \Gamma(n+1)$ equals $n!$.

9.3 The integral is analytic in s

Fix a point s_0 with $\operatorname{Re} s_0 > 0$ and a closed disc around it on which σ stays between two positive numbers σ_1 and σ_2. On that disc $|e^{-x}x^{s-1}| = e^{-x}x^{\sigma-1} \le e^{-x}(x^{\sigma_1-1} + x^{\sigma_2-1})$ for all $x > 0$ (the first term dominates for $x < 1$, the second for $x > 1$), a bound independent of s with a finite integral. For each fixed x the integrand is an entire function of s. The integral over any small triangle Δ in the disc may therefore be moved inside the x-integral (the dominating bound justifies exchanging the order of the two integrations), giving $\int_\Delta \Gamma(s)\,ds = \int_0^\infty e^{-x}\big[\int_\Delta x^{s-1}ds\big]dx = 0$, the inner integral vanishing by Cauchy's theorem. By Morera's theorem Γ is analytic at s_0. The same pattern of argument (a bound independent of the parameter, Cauchy inside, Morera outside) will be used for every parameter-dependent integral below and will be referred to as "the Morera argument".

9.4 Continuation of Γ to the whole plane; its poles; its residues

Rewrite (57) as $\Gamma(s) = \Gamma(s+1)/s$. The right side is defined and analytic for $\operatorname{Re} s > -1$, $s \ne 0$ (because $\Gamma(s+1)$ needs $\operatorname{Re}(s+1) > 0$), and agrees with the integral for $\operatorname{Re} s > 0$; by Fact 2 it is the continuation of Γ to $\operatorname{Re} s > -1$ minus the point 0. Apply the same rewriting $n+1$ times:

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)(s+2)\cdots(s+n)}, \tag{58}$$

valid for $\operatorname{Re} s > -(n+1)$ except at $s = 0, -1, \dots, -n$. Since n may be taken as large as desired, Γ is continued to the whole plane except the points $0, -1, -2, \dots$. At $s = -n$ the denominator of (58) has the simple factor $(s+n)$ and the other factors are nonzero, so $-n$ is a simple pole. Its residue is the limit of $(s+n)\Gamma(s)$ as $s \to -n$, which from (58) is

$$\operatorname{Res}_{s=-n}\Gamma(s) = \frac{\Gamma(1)}{(-n)(-n+1)\cdots(-1)} = \frac{1}{(-1)^n\,n!} = \frac{(-1)^n}{n!}, \tag{59}$$

the denominator being the product of the n negative numbers $-n, -n+1, \dots, -1$. So the residue of Γ at 0 is 1, at -1 is -1, at -2 is $\frac{1}{2}$. These values are used in Section 12.

9.5 The Beta integral and its relation to Γ

For $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$ define $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, convergent at $t = 0$ because $\operatorname{Re} a > 0$ and at $t = 1$ because $\operatorname{Re} b > 0$. The claim is

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{60}$$

Proof: write the product of two Gamma integrals as a double integral over the quarter-plane $x > 0$, $y > 0$,

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty e^{-x-y} x^{a-1} y^{b-1}\, dx\, dy,$$

and change variables to $r = x + y$ and $t = x/(x+y)$, that is, $x = rt$ and $y = r(1-t)$, with r running over $(0,\infty)$ and t over $(0,1)$. The Jacobian determinant of (x,y) with respect to (r,t) is $\begin{vmatrix} t & r \\ 1-t & -r \end{vmatrix} = -rt - r(1-t) = -r$, of modulus r, so $dx\,dy = r\,dr\,dt$. Then $x^{a-1}y^{b-1} = r^{a-1}t^{a-1}r^{b-1}(1-t)^{b-1}$ and $e^{-x-y} = e^{-r}$, so

$$\begin{aligned}\Gamma(a)\Gamma(b) &= \int_0^\infty \int_0^1 e^{-r} r^{a+b-2} t^{a-1} (1-t)^{b-1}\, r\, dt\, dr \\ &= \int_0^\infty e^{-r} r^{a+b-1} dr \cdot \int_0^1 t^{a-1}(1-t)^{b-1} dt = \Gamma(a+b)\, B(a,b). \qquad \blacksquare\end{aligned}$$

9.6 Euler's reflection formula, proved by a loop integral

The identity is

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin \pi s} \tag{61}$$

for every complex s that is not a whole number. It is proved first for $0 < \operatorname{Re} s < 1$ and then extended by Fact 2. In that strip both $\Gamma(s)$ and $\Gamma(1-s)$ are given by the integral, and (60) with $a = s$, $b = 1-s$ gives $\Gamma(s)\Gamma(1-s) = \Gamma(1)B(s,1-s) = \int_0^1 t^{s-1}(1-t)^{-s} dt$. Substitute $t = u/(1+u)$, so that $u = t/(1-t)$ runs over $(0,\infty)$ as t runs over $(0,1)$; then $1 - t = 1/(1+u)$ and $dt = du/(1+u)^2$ (quotient rule), so

$$t^{s-1}(1-t)^{-s} dt = \frac{u^{s-1}}{(1+u)^{s-1}} \cdot (1+u)^s \cdot \frac{du}{(1+u)^2} = \frac{u^{s-1}}{1+u}\, du,$$

the powers of $(1+u)$ combining as $-(s-1) + s - 2 = -1$. Hence

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{u^{s-1}}{1+u}\, du \qquad (0 < \operatorname{Re} s < 1). \tag{62}$$

This integral is evaluated by exactly the kind of loop integral Riemann uses in Section 11, so the evaluation is a rehearsal. Treat u as a complex variable and use the branch $u^{s-1} = e^{(s-1)\log u}$ with $0 < \arg u < 2\pi$, cut along the positive real axis (item 16 of Section 4). On the upper edge of the cut, $\arg u \to 0$ and $u^{s-1} \to x^{s-1}$ (real-positive branch); on the lower edge, $\arg u \to 2\pi$ and $u^{s-1} \to x^{s-1} e^{2\pi i(s-1)}$. Let the closed path K run from $u = \varepsilon$ to $u = R$ along the upper edge, counterclockwise around the circle $|u| = R$, back from R to ε along the lower edge, and clockwise around the circle $|u| = \varepsilon$. Inside K the integrand $u^{s-1}/(1+u)$ is analytic except at $u = -1$, where $1 + u$ has a simple zero with derivative 1; by the rule $\operatorname{Res}(g/h) = g(a)/h'(a)$ the residue there is the value of u^{s-1} at $u = -1 = e^{i\pi}$ (the argument π lies in the chosen interval), namely $e^{i\pi(s-1)}$. The residue theorem gives

$$\int_K \frac{u^{s-1}}{1+u}\, du = 2\pi i\, e^{i\pi(s-1)}.$$

The four pieces of K: the upper edge contributes $\int_\varepsilon^R x^{s-1} dx/(1+x)$; the lower edge, run from R down to ε, contributes $-e^{2\pi i(s-1)} \int_\varepsilon^R x^{s-1} dx/(1+x)$; on the large circle $|u^{s-1}| = R^{\sigma-1} e^{-t \arg u} \le R^{\sigma-1} e^{2\pi|t|}$ and $|1+u| \ge R - 1$, so the contribution is at most $2\pi R \cdot R^{\sigma-1} e^{2\pi|t|}/(R-1)$, which tends to 0 as $R \to \infty$ because $\sigma < 1$; on the small circle $|u^{s-1}| \le \varepsilon^{\sigma-1} e^{2\pi|t|}$ and $|1+u| \ge 1 - \varepsilon$, so the contribution is at most $2\pi\varepsilon \cdot \varepsilon^{\sigma-1} e^{2\pi|t|}/(1-\varepsilon)$, which tends to 0 as $\varepsilon \to 0$ because $\sigma > 0$. Letting $R \to \infty$ and $\varepsilon \to 0$,

$$\left(1 - e^{2\pi i(s-1)}\right) \int_0^\infty \frac{x^{s-1}}{1+x}\, dx = 2\pi i\, e^{i\pi(s-1)}.$$

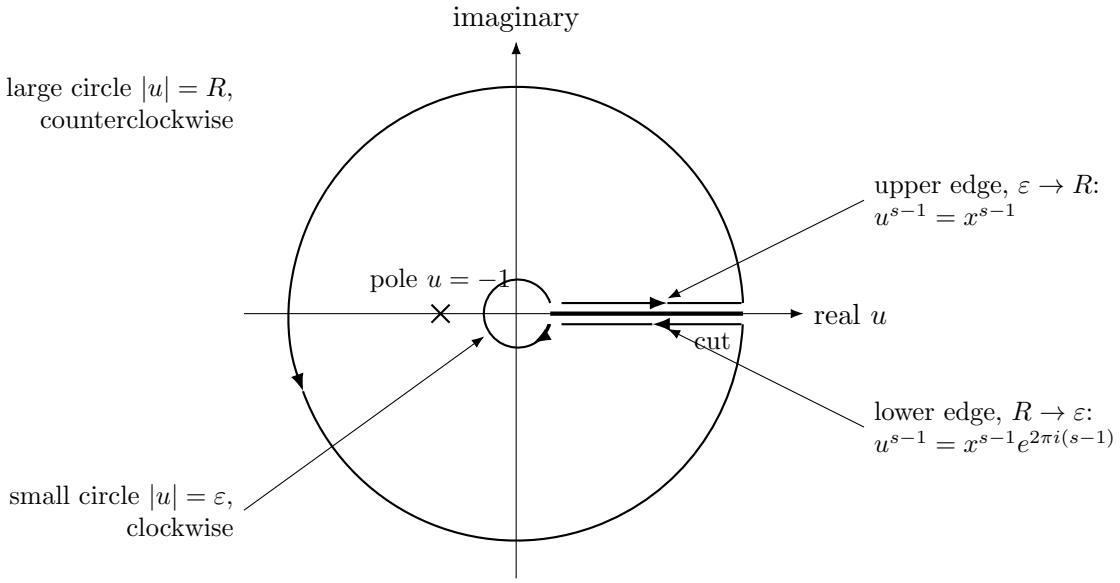


Figure 12: The closed path K of Section 9.6, drawn in the plane of the complex variable u. The integrand $u^{s-1}/(1+u)$ uses the branch with $0 < \arg u < 2\pi$, whose cut is the positive real axis (thick ray marked "cut"). K runs out along the upper edge of the cut from ε to R, around the large circle counterclockwise, back along the lower edge from R to ε, and around the small circle clockwise. The only pole inside is at $u = -1$, where $\arg u = \pi$. The two edges carry different values of u^{s-1}, differing by the factor $e^{2\pi i(s-1)}$, and that difference is what produces the sine in Euler's reflection formula.

Divide by $1 - e^{2\pi i(s-1)}$, then divide numerator and denominator by $e^{i\pi(s-1)}$:

$$\int_0^\infty \frac{x^{s-1}}{1+x}\,dx = \frac{2\pi i\, e^{i\pi(s-1)}}{1 - e^{2\pi i(s-1)}} = \frac{2\pi i}{e^{-i\pi(s-1)} - e^{i\pi(s-1)}} = \frac{2\pi i}{-2i\sin\pi(s-1)} = -\frac{\pi}{\sin\pi(s-1)} = \frac{\pi}{\sin\pi s},$$

using $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$ and $\sin(\pi s - \pi) = -\sin\pi s$. Together with (62) this proves (61) for $0 < \operatorname{Re} s < 1$. Both sides of (61) are meromorphic on the whole plane (the left by the continuation of Γ, the right because $\sin\pi s$ is entire with zeros only at the whole numbers), they agree on the open strip, so by Fact 2 they agree wherever both are defined. ■

Three consequences. (i) At $s = \frac12$: $\Gamma(\frac12)^2 = \pi/\sin(\pi/2) = \pi$, and $\Gamma(\frac12) > 0$ as an integral of a positive function, so $\Gamma(\frac12) = \sqrt{\pi}$. (ii) Γ has no zeros: at a non-whole-number s the product $\Gamma(s)\Gamma(1-s)$ equals $\pi/\sin\pi s \neq 0$, so neither factor is 0; at $s = 1, 2, 3, \ldots$ the value is $(s-1)! \neq 0$; at $s = 0, -1, -2, \ldots$ there are poles. (iii) The form Riemann uses in Section 11: multiplying (61) by $\sin\pi s/\Gamma(s)$ gives

$$\frac{1}{\Gamma(s)} = \frac{\Gamma(1-s)\sin\pi s}{\pi}, \qquad \text{so} \qquad \frac{1}{2\sin(\pi s)\,\Gamma(s)} = \frac{\Gamma(1-s)}{2\pi}. \tag{63}$$

9.7 Legendre's duplication formula, proved by the Beta integral

The identity is

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}}\,\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big). \tag{64}$$

Proof for $\operatorname{Re} s > 0$, then Fact 2. Put $a = b = s/2$ in (60): $B(\frac{s}{2}, \frac{s}{2}) = \Gamma(\frac{s}{2})^2/\Gamma(s)$, and by definition $B(\frac{s}{2}, \frac{s}{2}) = \int_0^1 \big(t(1-t)\big)^{s/2-1} dt$. Substitute $t = (1+v)/2$, so v runs from -1 to 1, $dt = dv/2$, and $t(1-t) = \frac{1+v}{2}\cdot\frac{1-v}{2} = \frac{1-v^2}{4}$:

$$B\Big(\frac{s}{2},\frac{s}{2}\Big) = \int_{-1}^1 \Big(\frac{1-v^2}{4}\Big)^{s/2-1}\frac{dv}{2} = \frac{4^{1-s/2}}{2}\int_{-1}^1 (1-v^2)^{s/2-1}dv = 2^{1-s}\cdot 2\int_0^1 (1-v^2)^{s/2-1}dv,$$

using $4^{1-s/2}/2 = 2^{2-s}/2 = 2^{1-s}$ and the evenness of the integrand to double the integral over $(0,1)$. Now substitute $w = v^2$, $v = \sqrt{w}$, $dv = dw/(2\sqrt{w}) = \frac{1}{2}w^{-1/2}dw$:

$$2\int_0^1 (1-v^2)^{s/2-1}dv = \int_0^1 w^{-1/2}(1-w)^{s/2-1}dw = B\Big(\frac{1}{2},\frac{s}{2}\Big) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2})}.$$

Combining: $\dfrac{\Gamma(\frac{s}{2})^2}{\Gamma(s)} = 2^{1-s}\dfrac{\sqrt{\pi}\,\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2})}$. Cancel one $\Gamma(\frac{s}{2})$ (nonzero by consequence (ii) above) and solve for $\Gamma(s)$: $\Gamma(s) = \dfrac{2^{s-1}}{\sqrt{\pi}}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})$, which is (64). Both sides are meromorphic on the plane and agree for $\operatorname{Re} s > 0$, so they agree everywhere. ■

The half-argument reflection formula. Riemann's symmetric form in Section 13 also uses (61) at the argument $(1-s)/2$: since $1 - \frac{1-s}{2} = \frac{1+s}{2}$,

$$\Gamma\Big(\frac{1-s}{2}\Big)\Gamma\Big(\frac{1+s}{2}\Big) = \frac{\pi}{\sin\frac{\pi(1-s)}{2}} = \frac{\pi}{\sin\left(\frac{\pi}{2}-\frac{\pi s}{2}\right)} = \frac{\pi}{\cos\frac{\pi s}{2}}, \tag{65}$$

using $\sin(\frac{\pi}{2}-\theta) = \cos\theta$. These are the "known properties of the function Π" that Riemann invokes without proof; he had them from Gauss's 1812 memoir (Section 2).

10 Step one: the Gamma integral replaces the series

The first operation in Riemann's paper, immediately after he writes down Euler's product, is to apply the integral (56) to each term of the series. In this section s is complex with $\operatorname{Re} s > 1$, n is a positive whole number, and x and u are real variables of integration.

10.1 One term of the series as an integral

Start from the definition (56), $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$, and substitute $x = nu$, where n is fixed. As x runs from 0 to ∞, so does $u = x/n$. The pieces transform as follows: $dx = n\,du$; $e^{-x} = e^{-nu}$; and $x^{s-1} = (nu)^{s-1} = e^{(s-1)(\log n + \log u)} = e^{(s-1)\log n}e^{(s-1)\log u} = n^{s-1}u^{s-1}$. Putting the three pieces in,

$$\Gamma(s) = \int_0^\infty e^{-nu}\,n^{s-1}u^{s-1}\,n\,du = n^{s-1}\cdot n\int_0^\infty e^{-nu}u^{s-1}\,du = n^s\int_0^\infty e^{-nu}u^{s-1}\,du, \tag{66}$$

the constants n^{s-1} and n having been taken outside the integral and combined as $n^{s-1}\cdot n = n^s$. Divide both sides by n^s and rename the variable of integration u back to x:

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty e^{-nx}x^{s-1}\,dx. \tag{67}$$

This is the equation Riemann writes first, in his notation $\Pi(s-1)/n^s = \int_0^\infty e^{-nx}x^{s-1}dx$. Its effect is to move the n out of the base of a power, where it cannot be summed, into the exponent of e^{-nx}, where the sum over n is a geometric series.

10.2 Summing over n

Write (67) for $n = 1, 2, \ldots, N$ and add the N equations. On the left the sum is $\Gamma(s)\big(1^{-s} + 2^{-s} + \cdots + N^{-s}\big) = \Gamma(s)S_N(s)$, with S_N the partial sum of (2) (now with complex s). On the right a sum of finitely many integrals is the integral of the sum:

$$\Gamma(s)\,S_N(s) = \int_0^\infty \Big(\sum_{n=1}^{N} e^{-nx}\Big)x^{s-1}\,dx. \tag{68}$$

The finite geometric sum inside has first term e^{-x} and ratio e^{-x}, so it equals $e^{-x}(1-e^{-Nx})/(1-e^{-x})$; multiplying numerator and denominator by e^x turns this into $(1-e^{-Nx})/(e^x-1)$. Hence

$$\Gamma(s)\,S_N(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx - \int_0^\infty \frac{e^{-Nx}x^{s-1}}{e^x-1}\,dx, \tag{69}$$

provided the first integral on the right converges, which is checked in the next paragraph. The second integral tends to 0 as $N\to\infty$: for every $x>0$ one has $e^x-1\ge x$ (the power series of e^x-1 is $x+x^2/2+\cdots$, all terms positive), so $1/(e^x-1)\le 1/x$, and therefore the modulus of the integrand is at most $e^{-Nx}x^{\sigma-1}/x = e^{-Nx}x^{\sigma-2}$; by (66) with $\sigma-1$ in place of s (allowed since $\sigma-1>0$), $\int_0^\infty e^{-Nx}x^{\sigma-2}dx = \Gamma(\sigma-1)/N^{\sigma-1}$, which tends to 0 as $N\to\infty$. On the left of (69), $S_N(s)\to\zeta(s)$. So

$$\Gamma(s)\,\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx \qquad (\mathrm{Re}\,s>1), \tag{70}$$

which is Riemann's second displayed equation, $\Pi(s-1)\zeta(s) = \int_0^\infty x^{s-1}dx/(e^x-1)$.

Where the integral converges, and where it does not. The modulus of the integrand is $x^{\sigma-1}/(e^x-1)$. For $x\ge 1$: $e^x\ge e>2$, so $e^x-1\ge e^x/2$ and $1/(e^x-1)\le 2e^{-x}$; the integrand is at most $2e^{-x}x^{\sigma-1}$, whose integral over $[1,\infty)$ is finite for every σ. For $0<x\le 1$: from the power series, $x\le e^x-1\le x(1+\frac12+\frac16+\frac{1}{24}+\cdots) = x(e-1)<2x$, so the integrand lies between $\frac12 x^{\sigma-2}$ and $x^{\sigma-2}$, and $\int_0^1 x^{\sigma-2}dx$ is finite exactly when $\sigma-2>-1$, that is, when $\sigma>1$. So the integral (70) converges exactly when $\mathrm{Re}\,s>1$, and the only obstacle to convergence for other s is the single endpoint $x=0$.

What has been gained. The series has disappeared. The whole dependence on s now sits in the one factor x^{s-1}, and the other factor $1/(e^x-1)$ has no s in it. In the series, the condition $\mathrm{Re}\,s>1$ was needed term by term, for every n at once. In the integral it is needed at one point of the path of integration, $x=0$. A path that produces the same number but keeps away from $x=0$ would remove the condition. Constructing that path is Step two.

10.3 Two facts about $1/(e^x-1)$ for later use

The Bernoulli expansion. Near $x=0$ the quotient $(e^x-1)/x = 1+\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{24}+\cdots$ is a power series with constant term 1, so its reciprocal $x/(e^x-1)$ is also a power series near 0. Its coefficients are by definition the Bernoulli numbers divided by factorials:

$$\frac{x}{e^x-1} = \sum_{k\ge 0}\frac{B_k}{k!}x^k = B_0 + B_1x + \frac{B_2}{2}x^2 + \frac{B_3}{6}x^3 + \cdots. \tag{71}$$

The coefficients are computed from the identity $\frac{x}{e^x-1}\cdot\frac{e^x-1}{x}=1$ by multiplying the two series and matching powers of x. The product of (71) with $\sum_{j\ge0}x^j/(j+1)!$ has, as coefficient of x^m, the sum of $\frac{B_k}{k!}\cdot\frac{1}{(m-k+1)!}$ over $k=0,\dots,m$; this must be 1 for $m=0$ and 0 for $m\ge1$. In turn: $m=0$ gives $B_0=1$. $m=1$ gives $B_1+\frac{B_0}{2}=0$, so $B_1=-\frac12$. $m=2$ gives $\frac{B_2}{2}+\frac{B_1}{2}+\frac{B_0}{6}=0$, so $\frac{B_2}{2}=\frac14-\frac16=\frac{1}{12}$ and $B_2=\frac16$. $m=3$ gives $\frac{B_3}{6}+\frac{B_2}{4}+\frac{B_1}{6}+\frac{B_0}{24}=0$, that is $\frac{B_3}{6}+\frac{1}{24}-\frac{1}{12}+\frac{1}{24}=\frac{B_3}{6}$, so $B_3=0$. $m=4$ gives $\frac{B_4}{24}+\frac{B_3}{12}+\frac{B_2}{12}+\frac{B_1}{24}+\frac{B_0}{120}=0$, that is $\frac{B_4}{24}+\frac{1}{72}-\frac{1}{48}+\frac{1}{120}=\frac{B_4}{24}+\frac{10-15+6}{720}=\frac{B_4}{24}+\frac{1}{720}=0$, so $B_4=-\frac{24}{720}=-\frac{1}{30}$. Continuing gives $B_5=0$, $B_6=\frac{1}{42}$, $B_7=0$, $B_8=-\frac{1}{30}$.

Every odd-index Bernoulli number from B_3 on is 0, for the following reason. The function $f(x)=\frac{x}{e^x-1}+\frac{x}{2}$ is even, that is, $f(-x)=f(x)$: multiplying numerator and denominator of $\frac{-x}{e^{-x}-1}$ by e^x gives $\frac{-xe^x}{1-e^x}=\frac{xe^x}{e^x-1}$, so $f(-x)=\frac{xe^x}{e^x-1}-\frac{x}{2}$, and $f(x)-f(-x)=\frac{x-xe^x}{e^x-1}+x=\frac{-x(e^x-1)}{e^x-1}+x=-x+x=0$. An even function has only even powers in its power series. The series of f is (71) with the x-term changed from B_1x to $(B_1+\frac12)x=0$; its coefficient of x^k for odd $k\ge3$ is $B_k/k!$, which must therefore be 0.

The expansion (71) converges for $|x| < 2\pi$: the reciprocal of a power series is analytic as far as the nearest point where the original series vanishes, and $e^x - 1$ vanishes at $x = \pm 2\pi i$ and at no nearer point (next paragraph). Dividing (71) by x,

$$\frac{1}{e^x - 1} = \sum_{k \geq 0} \frac{B_k}{k!} x^{k-1} = \frac{1}{x} - \frac{1}{2} + \frac{x}{12} - \frac{x^3}{720} + \cdots \qquad (0 < |x| < 2\pi), \tag{72}$$

which is the Laurent expansion (item 13 of Section 4) of $1/(e^x - 1)$ at 0: a simple pole with residue $B_0 = 1$. Section 12 reads the values of ζ at the negative whole numbers off this expansion.

The poles of $1/(e^x - 1)$ as a function of a complex variable. Now let x be complex. $e^x = 1$ exactly when $x = 2\pi i k$ for a whole number k (positive, negative or zero), because $e^{u+iv} = e^u(\cos v + i \sin v)$ equals 1 only when $e^u = 1$ and $\cos v = 1$, $\sin v = 0$. Near $x = 2\pi i k$ write $x = 2\pi i k + h$; then $e^x - 1 = e^h - 1 = h + \frac{h^2}{2} + \cdots$, a simple zero with derivative 1. So $1/(e^x - 1)$ is meromorphic on the whole plane with simple poles at the points $2\pi i k$, each of residue $g(a)/h'(a) = 1/1 = 1$. These poles are the source of the functional equation in Section 13.

In modern language, (70) says that $\Gamma(s)\zeta(s)$ is the Mellin transform of $1/(e^x - 1)$ (item 26 of Section 4); the name is Mellin's and dates from the 1890s, after Riemann, but the operation is Riemann's.

11 Step two: the loop integral around the origin gives the continuation to the whole plane

This is the step in which the analytic continuation is invented. Riemann's words: "If one now considers the integral $\int (-x)^{s-1} dx/(e^x - 1)$ from $+\infty$ to $+\infty$ taken in a positive sense around a domain which includes the value 0 but no other discontinuity of the integrand in its interior, then this is easily seen to be equal to $(e^{-\pi s i} - e^{\pi s i}) \int_0^\infty x^{s-1} dx/(e^x - 1)$, provided that, in the many-valued function $(-x)^{s-1} = e^{(s-1)\log(-x)}$, the logarithm of $-x$ is determined so as to be real for negative x." Every clause of this sentence is now made explicit. In this section x is a complex variable and $s = \sigma + it$ is any complex number.

11.1 The branch of $(-x)^{s-1}$

For every complex x not on the ray $x \geq 0$ (that is, not a non-negative real number) define

$$(-x)^{s-1} = \exp\big((s-1)\log(-x)\big), \qquad \log(-x) = \log|x| + i \arg(-x), \qquad -\pi < \arg(-x) < \pi. \tag{73}$$

Here $\arg(-x)$ is the angle of the point $-x$ measured from the positive real axis, taken in the open interval $(-\pi, \pi)$ (item 16 of Section 4). When x is a negative real number, $-x$ is a positive real number, its angle is 0, and $\log(-x)$ is the ordinary real logarithm; that is Riemann's normalisation "real for negative x". The point $-x$ lies on the negative real axis, where $\arg(-x)$ would have to jump from π to $-\pi$, exactly when x lies on the positive real axis; so the branch cut of (73) is the ray $x \geq 0$, and on the plane with that ray removed the function is analytic in x.

On the two sides of the cut the function takes different values. Take a point $x > 0$ and approach it from above, through the points $x + i\varepsilon$ with ε small and positive. Then $-(x + i\varepsilon) = -x - i\varepsilon$ is a point just below the negative real axis, whose angle in $(-\pi, \pi)$ is just above $-\pi$; as $\varepsilon \to 0$ the angle tends to $-\pi$. Approach the same x from below, through $x - i\varepsilon$: then $-x + i\varepsilon$ is just above the negative real axis, with angle tending to $+\pi$. Putting these angles into (73), and writing x^{s-1} for the real-positive power $e^{(s-1)\log x}$,

$$\begin{aligned} &\text{upper edge: } (-x)^{s-1} = e^{(s-1)(\log x - i\pi)} = x^{s-1} e^{-i\pi(s-1)}, \\ &\text{lower edge: } (-x)^{s-1} = e^{(s-1)(\log x + i\pi)} = x^{s-1} e^{+i\pi(s-1)}. \end{aligned} \tag{74}$$

The two values differ by the factor $e^{2\pi i(s-1)}$, as crossing a cut always produces.

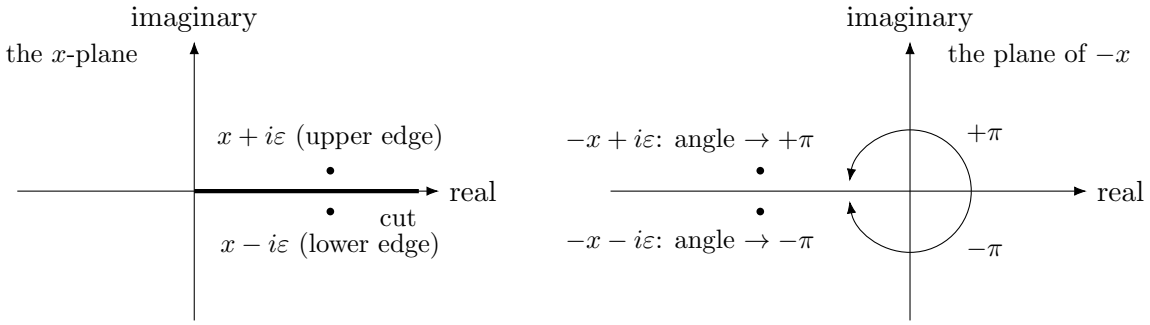


Figure 13: The branch (73) of $(-x)^{s-1}$. Left: the x-plane, with the cut along the ray $x \geq 0$ (thick) and two points just above and just below it. Right: the same two points after the sign change $x \mapsto -x$. The point $x + i\varepsilon$ above the cut becomes $-x - i\varepsilon$, just below the negative real axis, whose angle in $(-\pi, \pi)$ tends to $-\pi$; so $(-x)^{s-1} = x^{s-1}e^{-i\pi(s-1)}$ on the upper edge. The point $x - i\varepsilon$ below the cut becomes $-x + i\varepsilon$, just above the negative real axis, with angle tending to $+\pi$; so $(-x)^{s-1} = x^{s-1}e^{+i\pi(s-1)}$ on the lower edge. These are the two edge values (74).

11.2 The path, and the integral $I(s)$

Fix a radius δ with $0 < \delta < 2\pi$. Let C be the path that comes in from $+\infty$ along the upper edge of the cut to the point $x = \delta$, runs once counterclockwise around the circle $|x| = \delta$, and goes out along the lower edge of the cut back to $+\infty$ (Figure 14). The symbol $+\infty$ in Riemann's phrase "from $+\infty$ to $+\infty$" is the far right end of the positive real axis of the x-plane, that is, of the thick ray in the figure: the path starts there on the upper side of the cut, and it ends there on the lower side; the two ends are the same place on the axis but on opposite sides of the cut, where the integrand takes its two different edge values. Counterclockwise is Riemann's "positive sense", and it forces the inbound leg to be the upper edge: a counterclockwise turn that starts on the positive real axis moves first into the upper half-plane, so the circle is entered from above and left from below. The path meets no singularity of the integrand: $1/(e^x - 1)$ has its poles at $x = 2\pi ik$, and the circle of radius $\delta < 2\pi$ encloses the one at $k = 0$ and excludes all others; the branch factor is analytic off the cut, and the path runs along the edges of the cut without crossing it. Define

$$I(s) = \int_C \frac{(-x)^{s-1}}{e^x - 1}\, dx. \tag{75}$$

The value of $I(s)$ does not depend on δ in the range $0 < \delta < 2\pi$: two such paths with radii $\delta_1 < \delta_2$ differ by the closed curve consisting of the two circles and the two segments of edge between them, which bounds a region (the cut annulus) in which the integrand is analytic, so by Cauchy's theorem the integral around that closed curve is 0 and the two loop integrals are equal.

11.3 Claim 1: $I(s)$ is an entire function of s

First, the size of the branch factor on the two rays, where $x \geq \delta$ is real. Multiply out the exponent in (74), with $s - 1 = (\sigma - 1) + it$ and the sign $\mp$ for the upper and lower edge respectively:

$$(s-1)(\log x \mp i\pi) = (\sigma-1)\log x \mp i\pi(\sigma-1) + it\log x \mp i^2\pi t = \big[(\sigma-1)\log x \pm \pi t\big] + i\big[t\log x \mp \pi(\sigma-1)\big].$$

The modulus of e^{A+iB} for real A, B is e^A, so

$$\big|(-x)^{s-1}\big| = e^{(\sigma-1)\log x \pm \pi t} = x^{\sigma-1}e^{\pm\pi t} \leq x^{\sigma-1}e^{\pi|t|} \qquad (x > 0 \text{ on either edge}). \tag{76}$$

Now take any closed disc K in the s-plane; on it $\sigma \leq A$ and $|t| \leq T$ for some constants A, T. For s in K and $x \geq \delta$ on either ray, the modulus of the integrand of (75) is at most $e^{\pi T}x^{A-1}/(e^x - 1)$, and this function of x alone has a finite integral over $[\delta, \infty)$ (it is at most $2e^{\pi T}x^{A-1}e^{-x}$ for $x \geq 1$, and bounded on $[\delta, 1]$). On the circle, the integrand is a continuous function of the pair (x, s) on a compact set, hence bounded. For each fixed x on C the integrand is an entire function of s (an

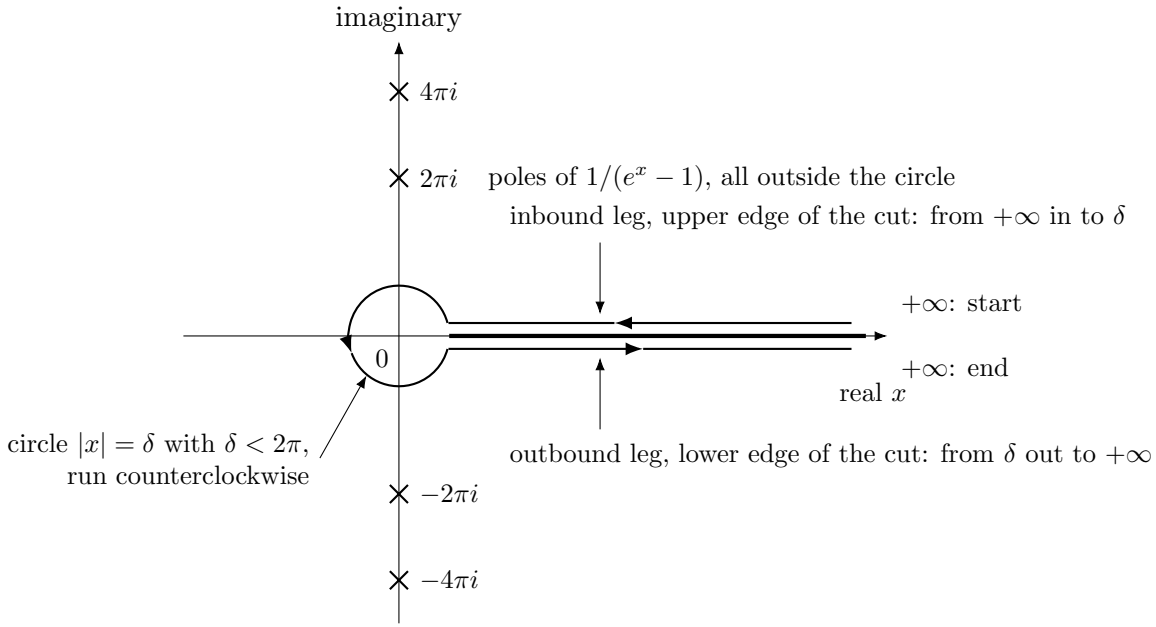


Figure 14: The loop C of (75), drawn in the plane of the complex variable x. "From $+\infty$ to $+\infty$" in Riemann's sentence means: the path begins at the far right end of the positive real axis, on the upper side of the cut (the cut is the thick ray $x \geq 0$), travels inward along the upper edge to the point $x = \delta$, goes once around the circle $|x| = \delta$ counterclockwise, and travels outward along the lower edge of the cut back to the far right end. The two legs lie on top of the same ray but carry the two different edge values (74) of $(-x)^{s-1}$. The crosses are the poles $x = 2\pi ik$ of $1/(e^x - 1)$; the choice $\delta < 2\pi$ keeps all of them except $x = 0$ outside the circle, and $x = 0$ itself is encircled without being touched.

exponential of a linear function of s times a constant). The Morera argument of Section 9 applies: for any small triangle Δ in K, $\int_\Delta I(s)\,ds = \int_C \big[\int_\Delta (-x)^{s-1} ds\big] \frac{dx}{e^x-1} = 0$, the exchange of the two integrations being justified by the bound independent of s, and the inner integral vanishing by Cauchy's theorem; so I is analytic on K. Since K was any disc, I is entire. ■

This is the whole reason for the loop: the path keeps the distance δ from $x = 0$, so the behaviour of x^{s-1} at the origin, the one place where the convergence of (70) depended on s, never enters.

11.4 Claim 2: for $\operatorname{Re} s > 1$, $I(s) = -2i \sin(\pi s)\, \Gamma(s)\, \zeta(s)$

Since $I(s)$ does not depend on δ, compute the three pieces of C and let $\delta \to 0$; in this computation $\sigma > 1$.

The inbound ray. It runs from $x = \infty$ down to $x = \delta$ along the upper edge, where by (74) the branch factor is $x^{s-1} e^{-i\pi(s-1)}$. An integral from ∞ to δ is minus the integral from δ to ∞:

$$\int_\infty^\delta \frac{x^{s-1} e^{-i\pi(s-1)}}{e^x - 1}\, dx = -\, e^{-i\pi(s-1)} \int_\delta^\infty \frac{x^{s-1}}{e^x - 1}\, dx. \tag{77}$$

The outbound ray. It runs from $x = \delta$ to ∞ along the lower edge, where the branch factor is $x^{s-1} e^{+i\pi(s-1)}$:

$$\int_\delta^\infty \frac{x^{s-1} e^{+i\pi(s-1)}}{e^x - 1}\, dx = +\, e^{+i\pi(s-1)} \int_\delta^\infty \frac{x^{s-1}}{e^x - 1}\, dx. \tag{78}$$

The two rays together. Adding (77) and (78) and factoring out the common integral,

$$\big(e^{i\pi(s-1)} - e^{-i\pi(s-1)}\big) \int_\delta^\infty \frac{x^{s-1}}{e^x - 1}\, dx = 2i \sin\big(\pi(s-1)\big) \int_\delta^\infty \frac{x^{s-1}}{e^x - 1}\, dx = -2i \sin(\pi s) \int_\delta^\infty \frac{x^{s-1}}{e^x - 1}\, dx, \tag{79}$$

using $e^{i\theta} - e^{-i\theta} = 2i \sin\theta$ and then $\sin(\pi s - \pi) = -\sin(\pi s)$.

The circle. On $|x| = \delta$ the modulus of $(-x)^{s-1}$ is $e^{(\sigma-1)\log\delta - t\arg(-x)} \leq \delta^{\sigma-1} e^{\pi|t|}$ (the same computation as (76) with a general angle in $(-\pi, \pi)$). Once $\delta \leq 1$, $|e^x - 1| \geq |x| - |x|^2/2 - \cdots \geq$

$\delta(1-\delta/2-\delta^2/6-\cdots)\geq\delta/2$, so $|1/(e^x-1)|\leq 2/\delta$. The circle has length $2\pi\delta$. So the circle contributes at most $2\pi\delta\cdot\delta^{\sigma-1}e^{\pi|t|}\cdot 2/\delta = 4\pi e^{\pi|t|}\delta^{\sigma-1}$, which tends to 0 as $\delta\to 0$ because $\sigma-1>0$.

Letting $\delta\to 0$. The integral in (79) tends to $\int_0^\infty x^{s-1}dx/(e^x-1)$, which converges for $\sigma>1$ and equals $\Gamma(s)\zeta(s)$ by (70); the circle contributes 0 in the limit; and $I(s)$ itself does not change. Hence

$$I(s) = -2i\sin(\pi s)\,\Gamma(s)\,\zeta(s) \qquad (\operatorname{Re} s>1). \qquad \blacksquare \tag{80}$$

Multiplying (80) by i and using $i\cdot(-2i)=2$ gives Riemann's equation exactly as he prints it,

$$2\sin(\pi s)\,\Pi(s-1)\,\zeta(s) = i\int_\infty^\infty \frac{(-x)^{s-1}\,dx}{e^x-1}, \tag{81}$$

his limits "∞ to ∞" being shorthand for the loop C, and his factor $(e^{-\pi si}-e^{\pi si})$ being $-2i\sin\pi s$.

11.5 Claim 3: equation (81) defines $\zeta(s)$ for every complex $s\neq 1$

Solve (80) for $\zeta(s)$ and replace $1/(2\sin(\pi s)\Gamma(s))$ by $\Gamma(1-s)/(2\pi)$, which is the form (63) of the reflection formula:

$$\zeta(s) = \frac{i\,I(s)}{2\sin(\pi s)\,\Gamma(s)} = \frac{i\,\Gamma(1-s)}{2\pi}\,I(s). \tag{82}$$

The right side is the product of the entire function $I(s)$ (Claim 1) and the meromorphic function $\Gamma(1-s)$, whose only singularities are simple poles where $1-s=0,-1,-2,\ldots$, that is, at $s=1,2,3,\ldots$. So the right side is a meromorphic function on the whole plane. It agrees with the series $\sum n^{-s}$ on the half-plane $\operatorname{Re} s>1$ by Claim 2. By Fact 2 of Section 8 it is therefore *the* analytic continuation of ζ, and (82) is the definition of $\zeta(s)$ at every complex s except possibly the points $1,2,3,\ldots$, which are examined one at a time in the next section. The outcome there: the pole at $s=1$ is real and all the others are cancelled by zeros of I. Riemann summarises: "This equation now gives the value of the function $\zeta(s)$ for all complex s and shows that it is single-valued and finite for all finite values of s except 1, and also that it vanishes when s equals a negative even integer."

11.6 Why the device works, said once more in words

The two values of $(-x)^{s-1}$ on the two sides of the cut are used as a tool. The loop integral equals the half-line integral of Step one multiplied by the difference of the two branch values, $2i\sin(\pi(s-1))$; so the half-line integral is recovered from the loop integral by one division. But the loop keeps away from $x=0$, so the loop integral converges and is analytic for every s, with no condition on $\operatorname{Re} s$. The division reintroduces difficulty only where the divisor $\sin(\pi s)\Gamma(s)$ vanishes, and those are isolated points, each of which can be examined separately. The path is now called the Hankel contour after Hermann Hankel (1839–1873), who applied it to the Gamma function itself in 1864; Riemann's use of it in 1859 is the earlier one.

12 Step three: the pole at $s=1$, the trivial zeros, and the values at the negative whole numbers, read off the loop integral

Everything in this section is obtained by evaluating $I(s)$ at a whole number $s=m$, where the branch factor becomes single-valued and the loop integral collapses to a residue. In this section m is a whole number, positive, zero or negative.

12.1 The loop integral at a whole number

When $s=m$, the two edge values (74) coincide: $e^{\mp i\pi(m-1)} = \cos(\pi(m-1)) \mp i\sin(\pi(m-1)) = (-1)^{m-1}$ in both cases, since the cosine of a whole multiple of π is ± 1 and the sine is 0. So the

inbound integral (77) and the outbound integral (78) are negatives of each other and cancel exactly, and only the circle remains:

$$I(m) = \oint_{|x|=\delta} \frac{(-x)^{m-1}}{e^x - 1}\,dx = 2\pi i \operatorname{Res}_{x=0} \frac{(-x)^{m-1}}{e^x - 1} = 2\pi i\,(-1)^{m-1} \operatorname{Res}_{x=0} \frac{x^{m-1}}{e^x - 1}, \tag{83}$$

by the residue theorem (item 14 of Section 4; the circle is run counterclockwise, and $(-x)^{m-1} = (-1)^{m-1}x^{m-1}$ for a whole number m). The residue is the coefficient of x^{-1} in the Laurent expansion of $x^{m-1}/(e^x - 1)$ at 0, which is obtained by multiplying (72) by x^{m-1}:

$$\frac{x^{m-1}}{e^x - 1} = \sum_{k\geq 0} \frac{B_k}{k!} x^{k+m-2}. \tag{84}$$

The exponent $k + m - 2$ equals -1 exactly when $k = 1 - m$. If $m \geq 2$ this would need $k \leq -1$, and the sum has no such term, so the residue is 0. If $m \leq 1$ the term $k = 1 - m$ exists and the residue is $B_{1-m}/(1 - m)!$. So

$$\operatorname{Res}_{x=0} \frac{x^{m-1}}{e^x - 1} = \begin{cases} \dfrac{B_{1-m}}{(1-m)!} & (m \leq 1), \\ 0 & (m \geq 2). \end{cases} \tag{85}$$

12.2 The whole numbers $m \geq 2$: no pole, and the value of the series is recovered

For $m \geq 2$ the residue is 0, so $I(m) = 0$. In the formula (82), $\Gamma(1 - s)$ has a simple pole at $s = m$ and $I(s)$ has a zero there. Since I is analytic, $I(s) = (s - m)\,\tilde{I}(s)$ with $\tilde{I}$ analytic near m, and since $\Gamma(1 - s)$ has a simple pole, $(s - m)\Gamma(1 - s)$ tends to a finite limit (the residue of $\Gamma(1 - s)$ at $s = m$, which is $-\operatorname{Res}_{u=1-m} \Gamma(u)$). So the product $\Gamma(1 - s)I(s) = \big[(s - m)\Gamma(1 - s)\big]\tilde{I}(s)$ tends to a finite limit as $s \to m$: the singularity of the right side of (82) at $s = m$ is removable. The value there is the value of the series: for s near m with $s \neq m$, the right side equals $\sum n^{-s}$ by Claim 2, and both the right side (after removing the singularity) and the series are continuous at m, so they agree at m. Nothing new happens at $s = 2, 3, 4, \ldots$, which is as it should be.

12.3 The point $s = 1$: a simple pole with residue exactly 1

For $m = 1$, (85) gives the residue $B_0/0! = 1$, so by (83) $I(1) = 2\pi i\,(-1)^0 \cdot 1 = 2\pi i$. Near $s = 1$ write $s = 1 + \varepsilon$ with ε small. By the recursion (57) in the form $\Gamma(u) = \Gamma(u + 1)/u$ with $u = -\varepsilon$, $\Gamma(1 - s) = \Gamma(-\varepsilon) = \Gamma(1 - \varepsilon)/(-\varepsilon)$, and $\Gamma(1 - \varepsilon) \to \Gamma(1) = 1$; so $\Gamma(1 - s)$ behaves like $-1/\varepsilon$. Also $I(1 + \varepsilon) \to I(1) = 2\pi i$ by continuity. Put both into (82):

$$\zeta(1 + \varepsilon) = \frac{i\,\Gamma(-\varepsilon)}{2\pi}\,I(1 + \varepsilon) \;\sim\; \frac{i}{2\pi} \cdot \Big(-\frac{1}{\varepsilon}\Big) \cdot 2\pi i = \frac{-i \cdot i \cdot 2\pi}{2\pi\varepsilon} = \frac{1}{\varepsilon} \qquad (\varepsilon \to 0), \tag{86}$$

using $i \cdot i = -1$. So $\zeta(s)$ behaves like $1/(s-1)$ near $s = 1$: a simple pole with residue exactly 1. This is the pole whose residue Euler's two comparisons (17) had already pinned to 1 for real s. Since I is entire and $\Gamma(1 - s)$ has no other poles than the ones just examined, $s = 1$ is the only singularity of ζ in the whole finite plane.

12.4 The negative whole numbers: the values Euler attached to divergent series

For $m = -n$ with n a non-negative whole number, (85) gives the residue $B_{n+1}/(n + 1)!$, and (83) gives $I(-n) = 2\pi i\,(-1)^{-n-1}B_{n+1}/(n + 1)!$, where $(-1)^{-n-1} = (-1)^{n+1}$ since the parity of $-n - 1$ and $n + 1$ is the same. In (82), $\Gamma(1 - s)$ at $s = -n$ is $\Gamma(1 + n) = n!$. Therefore

$$\begin{aligned} \zeta(-n) &= \frac{i\,n!}{2\pi} \cdot 2\pi i\,(-1)^{n+1} \frac{B_{n+1}}{(n+1)!} = i \cdot i \cdot (-1)^{n+1} \frac{n!}{(n+1)!} B_{n+1} \\ &= (-1) \cdot (-1)^{n+1} \frac{B_{n+1}}{n+1} = (-1)^n \frac{B_{n+1}}{n+1}, \end{aligned} \tag{87}$$

using $n!/(n+1)! = 1/(n+1)$. With the Bernoulli numbers computed in Section 10:

$$\zeta(0) = \frac{B_1}{1} = -\frac{1}{2}, \qquad \zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12}, \qquad \zeta(-2) = \frac{B_3}{3} = 0,$$
$$\zeta(-3) = -\frac{B_4}{4} = \frac{1}{120}, \qquad \zeta(-4) = \frac{B_5}{5} = 0, \qquad \zeta(-5) = -\frac{B_6}{6} = -\frac{1}{252}.$$

Convert to the alternating series through (41), $\eta(s) = (1 - 2^{1-s})\zeta(s)$, which now holds for every s because both sides are analytic and agree for $\operatorname{Re} s > 1$ (Fact 2): $\eta(0) = (1-2)(-\frac{1}{2}) = \frac{1}{2}$; $\eta(-1) = (1-4)(-\frac{1}{12}) = \frac{1}{4}$; $\eta(-2) = (1-8)\cdot 0 = 0$; $\eta(-3) = (1-16)\cdot\frac{1}{120} = -\frac{1}{8}$. These are exactly the four numbers Euler obtained in Section 7 by his summation rule. The consequence promised there has arrived: Euler's values at the negative whole numbers are now the values of a function defined by a convergent integral, and their agreement with his rule is a theorem about that function rather than a convention about divergent series.

12.5 The trivial zeros

For $n = 2k$ with $k \geq 1$, (87) gives $\zeta(-2k) = B_{2k+1}/(2k+1)$, and $B_{2k+1} = 0$ because every odd-index Bernoulli number from B_3 on vanishes (Section 10). So

$$\zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = 0. \tag{88}$$

These are the zeros Riemann mentions in the sentence quoted at the end of Section 11; they are called the trivial zeros, because their location is read off directly in this way. They are the only zeros of ζ on the real axis to the left of 0: at the odd negative whole numbers the value (87) is a nonzero fraction, and at every non-whole-number $s < 0$ the functional equation of the next section, $\zeta(s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(1-s)$, has every factor finite and nonzero: $(2\pi)^{-s}$ is a positive number, $\cos(\pi s/2)$ vanishes only at odd whole numbers, $\Gamma(s)$ is finite and nonzero away from the non-positive whole numbers (Section 9), and $\zeta(1-s) \geq 1$ because $1 - s > 1$.

12.6 Consistency with Euler's values at the positive even numbers

The functional equation of the next section, applied at $s = 2k$, reads

$$\zeta(1-2k) = 2(2\pi)^{-2k}\cos(k\pi)\,\Gamma(2k)\,\zeta(2k).$$

Here $\cos(k\pi) = (-1)^k$ and $\Gamma(2k) = (2k-1)!$. The left side is known from (87) with $n = 2k-1$: $\zeta(1-2k) = (-1)^{2k-1}B_{2k}/(2k) = -B_{2k}/(2k)$. Solve for $\zeta(2k)$:

$$\zeta(2k) = \frac{-B_{2k}/(2k)}{2(2\pi)^{-2k}(-1)^k(2k-1)!} = -\frac{B_{2k}}{2k}\cdot\frac{(2\pi)^{2k}}{2(-1)^k(2k-1)!} = (-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2\,(2k)!}, \tag{89}$$

using $2k\cdot(2k-1)! = (2k)!$ and $-1/(-1)^k = (-1)^{k+1}$. This is Euler's formula (39); for $k = 1$ it reads $\zeta(2) = (+1)\cdot\frac{4\pi^2\cdot\frac{1}{6}}{2\cdot 2} = \frac{\pi^2}{6}$. So Riemann's single definition (82) reproduces every value Euler had, whether Euler reached it by a convergent series or by his summation rule.

13 Step four: moving the contour onto the poles $x = \pm 2\pi i n$ gives the functional equation

Riemann's next paragraph proves Euler's 1749 relation by pushing the loop C outward until it wraps around the poles of $1/(e^x - 1)$. His words: "If the real part of s is negative, then, instead of being taken in a positive sense around the specified domain, this integral can also be taken in a negative sense around that domain containing all the remaining complex quantities, since the integral taken through values of infinitely large modulus is then infinitely small. However, in the interior of this domain, the function to be integrated has discontinuities only where x becomes equal to a whole

multiple of $\pm 2\pi i$, and the integral is thus equal to the sum of the integrals taken in a negative sense around these values. But the integral around the value $n2\pi i$ is $(-n2\pi i)^{s-1}(-2\pi i)$." The argument is now written out. Throughout, $s = \sigma + it$ with $\sigma < 0$, n is a positive whole number, and N is a positive whole number that will tend to infinity.

13.1 The region and its boundary

Put $R = (2N+1)\pi$. Consider the region A_N of points x with $\delta < |x| < R$, cut along the positive real axis: an annulus with a slit. Its boundary, run so that A_N lies on the left of the direction of travel, consists of four pieces in order: the large circle $|x| = R$ counterclockwise; the lower edge of the slit from $x = R$ inward to $x = \delta$; the small circle $|x| = \delta$ clockwise; the upper edge of the slit from $x = \delta$ outward to $x = R$. (To check the direction: walking counterclockwise on the outer circle keeps the inside on the left; arriving at the slit from below and walking inward along its lower edge keeps the annulus on the left, since the annulus lies below the lower edge; and so on.) The last three pieces are the loop C of Section 11, truncated at $|x| = R$, run backwards; call the truncated loop C_R, run in its original direction. Then the boundary integral is $\oint_{|x|=R} - \int_{C_R}$.

Inside A_N the integrand $(-x)^{s-1}/(e^x - 1)$ is analytic except at the poles $x = 2\pi ik$ of $1/(e^x - 1)$ (Section 10) that satisfy $\delta < |2\pi ik| < R$, that is, $0 < |k| \le N$, because $|2\pi ik| = 2\pi|k|$ and the circle of radius $(2N+1)\pi$ lies between $2\pi N$ and $2\pi(N+1)$. The branch factor contributes no singularity inside A_N, since its cut has been removed. The residue theorem (item 14 of Section 4) therefore gives

$$\oint_{|x|=R} \frac{(-x)^{s-1}}{e^x - 1}\,dx - \int_{C_R} \frac{(-x)^{s-1}}{e^x - 1}\,dx = 2\pi i \sum_{n=1}^{N} \Big[\mathrm{Res}_{x=2\pi in} \frac{(-x)^{s-1}}{e^x - 1} + \mathrm{Res}_{x=-2\pi in} \frac{(-x)^{s-1}}{e^x - 1} \Big]. \quad (90)$$

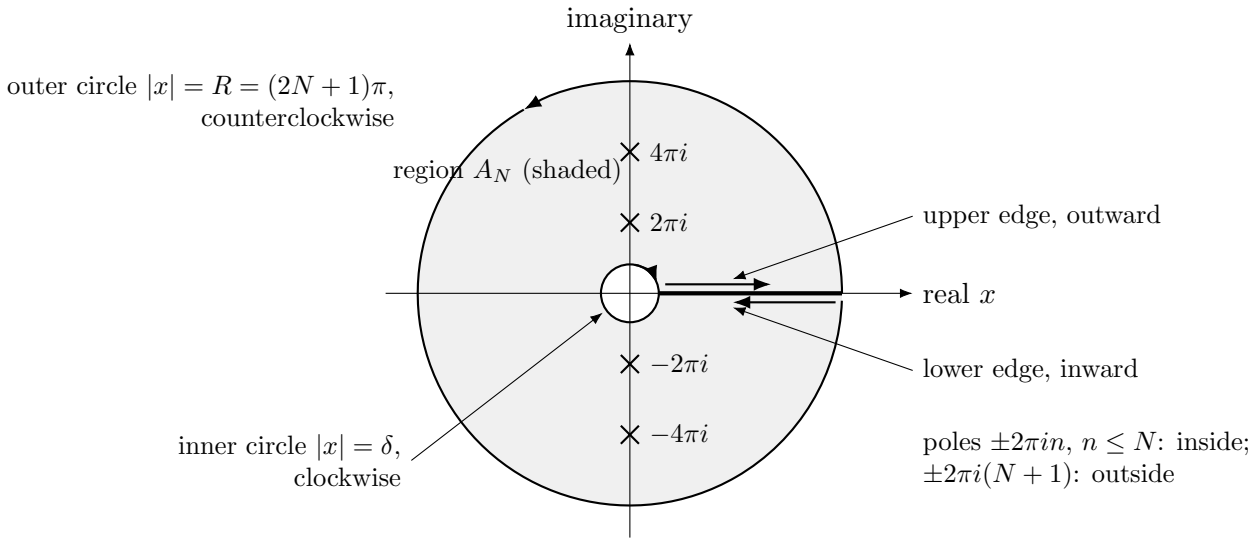


Figure 15: The region A_N of Section 13, for $N = 2$: the annulus $\delta < |x| < (2N+1)\pi$ with the positive real axis slit out. Its boundary, run with the region on the left, is the outer circle counterclockwise, the lower edge of the slit inward, the inner circle clockwise, and the upper edge outward; the last three pieces are the loop C of Figure 14 run backwards and cut off at $|x| = R$. The poles $\pm 2\pi in$ with $n \le N$ lie inside; the residue theorem turns the boundary integral into their residues, and the outer circle's contribution vanishes as $N \to \infty$ when $\mathrm{Re}\, s < 0$.

13.2 The residues

At $x = 2\pi in$ the denominator $e^x - 1$ has a simple zero with derivative $e^{2\pi in} = 1$ (Section 10), and the numerator $(-x)^{s-1}$ is analytic there. By the rule $\mathrm{Res}(g/h) = g(a)/h'(a)$ of item 13, the residue is the value of $(-x)^{s-1}$ at $x = 2\pi in$ divided by 1. To evaluate it, find the angle of $-x$ in the interval

$(-\pi, \pi)$. For $x = 2\pi i n$, the point $-x = -2\pi i n$ lies on the negative imaginary axis, at angle $-\pi/2$; so $\log(-x) = \log(2\pi n) - i\pi/2$ and

$$\mathrm{Res}_{x=2\pi i n} = \exp\big((s-1)(\log(2\pi n) - i\pi/2)\big) = (2\pi n)^{s-1} e^{-i\pi(s-1)/2}.$$

For $x = -2\pi i n$, the point $-x = 2\pi i n$ lies on the positive imaginary axis, at angle $+\pi/2$; so

$$\mathrm{Res}_{x=-2\pi i n} = (2\pi n)^{s-1} e^{+i\pi(s-1)/2}.$$

Their sum is

$$(2\pi n)^{s-1}\Big(e^{-i\pi(s-1)/2} + e^{+i\pi(s-1)/2}\Big) = (2\pi n)^{s-1} \cdot 2\cos\frac{\pi(s-1)}{2} = (2\pi n)^{s-1} \cdot 2\sin\frac{\pi s}{2}, \quad (91)$$

using $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and then $\cos\left(\frac{\pi s}{2} - \frac{\pi}{2}\right) = \sin\frac{\pi s}{2}$. In Riemann's notation the two exponentials are $(-i)^{s-1}$ and i^{s-1} (since $-i = e^{-i\pi/2}$ and $i = e^{i\pi/2}$), and his "integral taken in a negative sense around $n2\pi i$", namely $(-n2\pi i)^{s-1}(-2\pi i)$, is $-2\pi i$ times the first residue, the minus sign being the clockwise direction.

13.3 The large circle contributes nothing in the limit

Two estimates are needed on the circle $|x| = R = (2N+1)\pi$.

A lower bound for $|e^x - 1|$. Write $x = u + iv$ with u, v real, so $e^x = e^u(\cos v + i\sin v)$. Case $|u| \geq 1$: the modulus of e^x is e^u, which is either $\geq e$ or $\leq 1/e$, so $|e^x - 1| \geq \big||e^x| - 1\big| \geq \min(e-1,\, 1-1/e) = 1 - 1/e$. Case $|u| < 1$: then $v^2 = R^2 - u^2 > R^2 - 1$, so $|v|$ lies between $\sqrt{R^2-1}$ and R, an interval of length $R - \sqrt{R^2-1} = 1/(R + \sqrt{R^2-1}) < 1/R \leq 1/(3\pi) < 0.11$. Thus $|v|$ is within 0.11 of the odd multiple $(2N+1)\pi$ of π, where the cosine is -1; the cosine changes by at most 0.11 over that distance (its derivative is bounded by 1), so $\cos v \leq -0.89$. Then the real part of $e^x - 1$ is $e^u \cos v - 1 \leq -0.89 e^u - 1 < -1$, and the modulus of a complex number is at least the modulus of its real part, so $|e^x - 1| > 1$. In both cases

$$|e^x - 1| \geq 1 - \frac{1}{e} > 0.63 \qquad \text{on } |x| = (2N+1)\pi, \text{ for every } N \geq 1. \quad (92)$$

An upper bound for $|(-x)^{s-1}|$. With $\phi = \arg(-x) \in (-\pi, \pi)$, the computation of (76) with a general angle gives $|(-x)^{s-1}| = e^{(\sigma-1)\log R - t\phi} \leq R^{\sigma-1} e^{\pi|t|}$.

The estimate. The integral of a function along a path has modulus at most the length of the path times the maximum modulus of the function on it. The circle has length $2\pi R$. So

$$\Big|\oint_{|x|=R} \frac{(-x)^{s-1}}{e^x - 1}\, dx\Big| \leq 2\pi R \cdot \frac{R^{\sigma-1} e^{\pi|t|}}{1 - 1/e} = \frac{2\pi e^{\pi|t|}}{1-1/e}\, R^{\sigma}. \quad (93)$$

Since $\sigma < 0$, $R^\sigma = 1/R^{|\sigma|}$ tends to 0 as $R \to \infty$. This is the only place where the sign of $\mathrm{Re}\, s$ is used, and it is why Riemann states the hypothesis "if the real part of s is negative": for $\sigma \geq 0$ the right side of (93) does not tend to 0 and the contour cannot be pushed to infinity.

13.4 The limit $N \to \infty$

Let $N \to \infty$ in (90). The large-circle integral tends to 0 by (93). The truncated loop integral $\int_{C_R}$ tends to $I(s)$, because the two ray integrals in (75) converge (Claim 1 of Section 11) and C_R differs from C only by the parts of the rays beyond R. The residue sum, by (91), tends to $\sum_{n\geq 1} (2\pi n)^{s-1} \cdot 2\sin(\pi s/2)$, which converges absolutely because $|(2\pi n)^{s-1}| = (2\pi n)^{\sigma-1}$ and $\sigma - 1 < -1$. Hence

$$-I(s) = 2\pi i \cdot 2\sin\frac{\pi s}{2} \sum_{n=1}^{\infty} (2\pi n)^{s-1} = 4\pi i\,(2\pi)^{s-1} \sin\frac{\pi s}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} = 4\pi i\,(2\pi)^{s-1} \sin\frac{\pi s}{2}\, \zeta(1-s), \quad (94)$$

where $(2\pi n)^{s-1} = (2\pi)^{s-1}n^{s-1} = (2\pi)^{s-1}/n^{1-s}$, and $\sum n^{-(1-s)}$ is Euler's convergent series at the point $1-s$, whose real part $1-\sigma$ is greater than 1. Multiply (94) by $-i$ and use $-i\cdot 4\pi i = 4\pi$ and (81) (which says $2\sin(\pi s)\Gamma(s)\zeta(s) = iI(s)$, and $iI(s) = -i\cdot(-I(s))$):

$$2\sin(\pi s)\,\Gamma(s)\,\zeta(s) = 4\pi(2\pi)^{s-1}\sin\frac{\pi s}{2}\,\zeta(1-s) = (2\pi)^s\cdot 2\sin\frac{\pi s}{2}\cdot\zeta(1-s) \qquad (\operatorname{Re} s<0), \tag{95}$$

using $4\pi(2\pi)^{s-1} = 2\cdot 2\pi\cdot(2\pi)^{s-1} = 2(2\pi)^s$. This is Riemann's displayed equation, $2\sin\pi s\,\Pi(s-1)\zeta(s) = (2\pi)^s\sum n^{s-1}\big((-i)^{s-1}+i^{s-1}\big)$.

13.5 The functional equation

Use the double-angle identity $\sin(\pi s) = 2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}$ on the left of (95):

$$2\cdot 2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}\,\Gamma(s)\,\zeta(s) = (2\pi)^s\cdot 2\sin\frac{\pi s}{2}\cdot\zeta(1-s).$$

Divide both sides by $2\sin\frac{\pi s}{2}$, which is nonzero when s is not an even whole number:

$$2\cos\frac{\pi s}{2}\,\Gamma(s)\,\zeta(s) = (2\pi)^s\,\zeta(1-s), \qquad \text{that is,} \qquad \zeta(1-s) = 2\,(2\pi)^{-s}\cos\frac{\pi s}{2}\,\Gamma(s)\,\zeta(s). \tag{96}$$

This has been proved for every s with $\operatorname{Re} s<0$ that is not an even whole number; at the even negative whole numbers both sides are continuous, so it holds there too. Both sides of (96) are meromorphic functions on the whole plane: the left side is the continued ζ of Section 11 evaluated at $1-s$; the right side is the continued ζ at s times $\Gamma(s)$ times entire functions. Two meromorphic functions that agree on the open half-plane $\operatorname{Re} s<0$ agree wherever both are defined, by Fact 2. So (96) holds for every complex s, and it is, letter for letter, the relation (50) that Euler conjectured in 1749. ■

Consequence for the zeros. Take $\operatorname{Re} s>1$ on the right side of (96). There $\zeta(s)\neq 0$ by (54); $\Gamma(s)$ is finite and nonzero (Section 9); $(2\pi)^{-s}$ is nonzero; and $\cos(\pi s/2)$ vanishes exactly when $\pi s/2$ is an odd multiple of $\pi/2$, that is, when s is an odd whole number $2k+1$ with $k\geq 1$, for which $1-s=-2k$. So $\zeta(1-s)\neq 0$ for $\operatorname{Re} s>1$ except when $1-s$ is a negative even number: the function has no zeros in the half-plane $\operatorname{Re} s<0$ other than the trivial ones found in Section 12. Every other zero of ζ therefore lies in the closed strip $0\leq\operatorname{Re} s\leq 1$. The product formula controls $\operatorname{Re} s>1$; the functional equation carries that control to $\operatorname{Re} s<0$; the strip is what is left.

13.6 The symmetric form

Riemann adds that "through the use of known properties of the function Π" the relation can be put in the form: $\Pi(s/2-1)\pi^{-s/2}\zeta(s)$ is unchanged when s is replaced by $1-s$. The two known properties are Legendre's duplication formula (64) and the half-argument reflection formula (65), both proved in Section 9:

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}}\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big), \qquad \Gamma\Big(\frac{1-s}{2}\Big)\Gamma\Big(\frac{1+s}{2}\Big) = \frac{\pi}{\cos\frac{\pi s}{2}}.$$

Substitute the first into (96). The constants collect as $2\cdot(2\pi)^{-s}\cdot 2^{s-1}\pi^{-1/2} = 2\cdot 2^{-s}\pi^{-s}\cdot 2^{s-1}\pi^{-1/2} = 2^{1-s+s-1}\pi^{-s-1/2} = \pi^{-s-1/2}$, so

$$\zeta(1-s) = \pi^{-s-1/2}\cos\frac{\pi s}{2}\,\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big)\zeta(s).$$

Now replace $\cos\frac{\pi s}{2}$ by $\pi\big/\big(\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})\big)$ from the second property. The factor $\Gamma(\frac{s+1}{2})$ in the numerator cancels the same factor $\Gamma(\frac{1+s}{2})$ in the denominator, and $\pi^{-s-1/2}\cdot\pi = \pi^{1/2-s}$:

$$\zeta(1-s) = \frac{\pi^{1/2-s}\,\Gamma(\frac{s}{2})\,\zeta(s)}{\Gamma(\frac{1-s}{2})}.$$

Multiply both sides by $\pi^{-(1-s)/2}\Gamma(\frac{1-s}{2})$. On the right the power of π becomes $\pi^{1/2-s-(1-s)/2} = \pi^{1/2-s-1/2+s/2} = \pi^{-s/2}$. Hence

$$\pi^{-\frac{1-s}{2}}\,\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s) = \pi^{-\frac{s}{2}}\,\Gamma\Big(\frac{s}{2}\Big)\zeta(s). \tag{97}$$

The function $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ therefore satisfies $\Lambda(1-s) = \Lambda(s)$: it takes the same value at two points placed symmetrically about $s = \frac{1}{2}$. That is Riemann's statement. The consequence he draws from it is the subject of the next two sections: the factor $\Gamma(s/2)$ suggested a second, more direct route to the same function through Jacobi's theta series, and the symmetry about $s = \frac{1}{2}$ singles out the line $\operatorname{Re} s = \frac{1}{2}$ on which he conjectured all the remaining zeros lie.

14 Step five: the theta-series route (Jacobi) and the second derivation of the continuation

Riemann then gives a second construction of the same function. His words: "This property of the function induced me to introduce, in place of $\Pi(s-1)$, the integral $\Pi(s/2-1)$ into the general term of the series $\sum 1/n^s$, whereby one obtains a very convenient expression for the function $\zeta(s)$." The operation is that of Step one with $\Gamma(s/2)$ in place of $\Gamma(s)$ and $e^{-n^2\pi x}$ in place of e^{-nx}. In this section x, u, t and v are real variables, n and m are whole numbers, and $s = \sigma + it$ is complex; the letter t is used both for $\operatorname{Im} s$ and, in the Poisson-summation paragraph only, for a real integration variable, the context making clear which.

14.1 Two preliminary facts

The Gaussian integral. For $a > 0$, $\int_{-\infty}^{\infty} e^{-at^2}dt = \sqrt{\pi/a}$. Proof: call the integral G; then $G^2 = \iint e^{-a(t^2+v^2)}dt\,dv$ over the whole plane; in polar coordinates $t = r\cos\phi$, $v = r\sin\phi$, $dt\,dv = r\,dr\,d\phi$, this is $\int_0^{2\pi} d\phi \int_0^{\infty} e^{-ar^2} r\,dr = 2\pi\cdot\Big[-\frac{e^{-ar^2}}{2a}\Big]_0^{\infty} = 2\pi\cdot\frac{1}{2a} = \frac{\pi}{a}$, and $G > 0$, so $G = \sqrt{\pi/a}$. With $a = \pi x$: $\int_{-\infty}^{\infty} e^{-\pi x t^2}dt = x^{-1/2}$.

A bound for ψ near 0. Riemann's $\psi(x) = \sum_{n\ge1} e^{-n^2\pi x}$, for $x > 0$. The function $u \mapsto e^{-u^2\pi x}$ decreases for $u \ge 0$, so $e^{-n^2\pi x} \le \int_{n-1}^{n} e^{-u^2\pi x}du$ (the same comparison as (4)). Adding over $n \ge 1$ joins the integrals into one over $[0,\infty)$, which is half the Gaussian integral:

$$\psi(x) \le \int_0^{\infty} e^{-u^2\pi x}du = \frac{1}{2\sqrt{x}} \qquad (x > 0). \tag{98}$$

For $x \ge 1$ there is a sharper bound: since $n^2 \ge n$, $e^{-n^2\pi x} \le e^{-n\pi x}$, and the geometric series gives $\psi(x) \le e^{-\pi x}/(1-e^{-\pi x}) \le e^{-\pi x}/(1-e^{-\pi}) < 1.05\,e^{-\pi x}$.

14.2 The half-argument Gamma integral, term by term

In $\Gamma(s/2) = \int_0^{\infty} e^{-u}u^{s/2-1}du$ (valid for $\operatorname{Re} s > 0$) substitute $u = n^2\pi x$ for a fixed n; as u runs over $(0,\infty)$ so does x. Then $du = n^2\pi\,dx$ and $u^{s/2-1} = (n^2\pi)^{s/2-1}x^{s/2-1}$, so

$$\Gamma\Big(\frac{s}{2}\Big) = \int_0^{\infty} e^{-n^2\pi x}(n^2\pi)^{s/2-1}x^{s/2-1}\,n^2\pi\,dx = (n^2\pi)^{s/2}\int_0^{\infty} e^{-n^2\pi x}x^{s/2-1}\,dx,$$

the powers of $n^2\pi$ combining as $(n^2\pi)^{s/2-1}\cdot(n^2\pi)^1 = (n^2\pi)^{s/2}$. Since $(n^2\pi)^{s/2} = n^s\pi^{s/2}$, dividing by it gives

$$\frac{\Gamma(s/2)}{\pi^{s/2}\,n^s} = \int_0^{\infty} e^{-n^2\pi x}\,x^{s/2-1}\,dx \qquad (\operatorname{Re} s > 0), \tag{99}$$

which is Riemann's $(1/n^s)\Pi(s/2-1)\pi^{-s/2} = \int_0^{\infty} e^{-nn\pi x}x^{s/2-1}dx$.

14.3 Summing over n

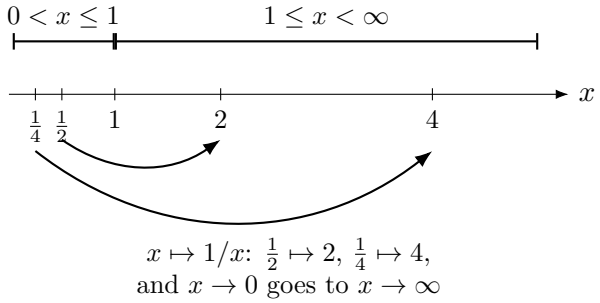
Now let $\operatorname{Re} s > 1$. Add (99) over $n = 1, \dots, N$; a finite sum of integrals is the integral of the finite sum $\psi_N(x) = \sum_{n=1}^{N} e^{-n^2\pi x}$:

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big) S_N(s) = \int_0^\infty \psi_N(x)\, x^{s/2-1}\, dx. \tag{100}$$

The claim is that as $N \to \infty$ the right side tends to $\int_0^\infty \psi(x) x^{s/2-1} dx$, that is, that $\int_0^\infty \big(\psi(x) - \psi_N(x)\big) x^{s/2-1} dx \to 0$. The difference $\psi - \psi_N = \sum_{n>N} e^{-n^2\pi x}$ is positive, so it suffices to bound the integral of $(\psi - \psi_N) x^{\sigma/2-1}$. Split it at $x = 1$ and, on $(0,1)$, at a small ε. On $(0, \varepsilon)$: by (98), $\psi - \psi_N \le \psi \le \frac{1}{2} x^{-1/2}$, so the integral is at most $\frac{1}{2}\int_0^\varepsilon x^{(\sigma-3)/2} dx = \frac{1}{2} \cdot \frac{\varepsilon^{(\sigma-1)/2}}{(\sigma-1)/2} = \frac{\varepsilon^{(\sigma-1)/2}}{\sigma-1}$, finite because $\sigma > 1$ and as small as desired by choice of ε. On $[\varepsilon, 1]$: $e^{-n^2\pi x} \le e^{-n\pi\varepsilon}$, so $\psi - \psi_N \le \sum_{n>N} e^{-n\pi\varepsilon} = e^{-(N+1)\pi\varepsilon}/(1 - e^{-\pi\varepsilon})$, which tends to 0 as $N \to \infty$ with ε fixed, and the integral of $x^{\sigma/2-1}$ over $[\varepsilon, 1]$ is finite. On $[1, \infty)$: $\psi - \psi_N \le \sum_{n>N} e^{-n\pi x} \le 1.05 e^{-(N+1)\pi x}$, so the integral is at most $1.05 \int_1^\infty e^{-(N+1)\pi x} x^{\sigma/2-1} dx \le 1.05 e^{-N\pi} \int_1^\infty e^{-\pi x} x^{\sigma/2-1} dx$, which tends to 0. Choosing ε first and then N makes the whole integral as small as desired. Meanwhile the left side of (100) tends to $\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Therefore

$$\pi^{-s/2}\,\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \int_0^\infty \psi(x)\, x^{s/2-1}\, dx \qquad (\operatorname{Re} s > 1). \tag{101}$$

As in Step one, the restriction $\operatorname{Re} s > 1$ comes only from the endpoint $x = 0$: there $\psi(x)$ is of size $x^{-1/2}$ (the bound (98) is sharp, as Jacobi's identity below shows), so the integrand is of size $x^{(s-3)/2}$, integrable at 0 exactly when $(\sigma - 3)/2 > -1$, that is, $\sigma > 1$. At $x = \infty$ the integrand decays like $e^{-\pi x}$ for every s.



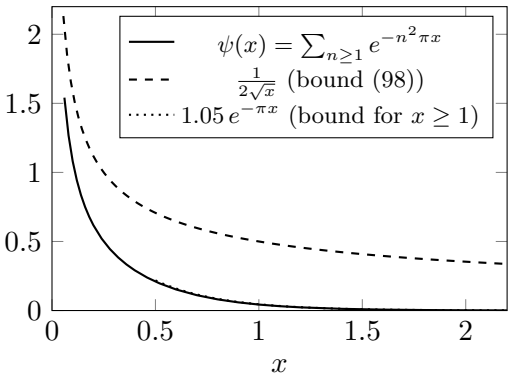


Figure 16: Left: the splitting of the integral (101) at $x = 1$, and the substitution $x \mapsto 1/x$ that folds the short piece $(0, 1]$ onto the long piece $[1, \infty)$, sending $\frac{1}{2}$ to 2, $\frac{1}{4}$ to 4, and $x \to 0$ to $x \to \infty$. Right: Riemann's $\psi(x)$ (solid) with its two bounds: near 0 it is about $\frac{1}{2\sqrt{x}}$ (dashed), which is why $\int_0^1 \psi(x) x^{s/2-1} dx$ needs $\operatorname{Re} s > 1$; for $x \ge 1$ it is below $1.05 e^{-\pi x}$ (dotted), which is why $\int_1^\infty$ converges for every s. Jacobi's identity converts the first behaviour into the second.

14.4 Jacobi's transformation, proved by Poisson's summation formula

The identity Riemann cites as "Jacobi, Fund. S. 184" is

$$2\psi(x) + 1 = x^{-1/2}\big(2\psi(1/x) + 1\big) \qquad (x > 0), \tag{102}$$

or, with the symmetric series $\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = 1 + 2\psi(x)$ (the terms n and $-n$ being equal), $\theta(x) = x^{-1/2}\theta(1/x)$, equivalently $\theta(1/x) = \sqrt{x}\,\theta(x)$. Jacobi (1804–1851) obtained it in 1829 from the transformation theory of elliptic functions. The proof given here uses the summation formula of Poisson (1781–1840), published in 1823, and takes two steps.

Step (a): Poisson's formula for the Gaussian. Fix $x > 0$ and let $f(t) = e^{-\pi x t^2}$ for real t. Form the function $F(t) = \sum_{n=-\infty}^{\infty} f(t+n)$. For t in a bounded interval, say $|t| \le 1$, and $|n| \ge 2$, one has $|t+n| \ge |n|-1$, so $f(t+n) \le e^{-\pi x(|n|-1)^2}$, and the sum of these over n is finite; by Weierstrass's test the series converges uniformly on bounded sets, and the same bound (with polynomial factors) applies to the series of derivatives of any order. So F is infinitely differentiable, and it has period 1: replacing t by $t+1$ just shifts the index n by one. A function with period 1 that is infinitely differentiable equals its Fourier series at every point:

$$F(t) = \sum_{m=-\infty}^{\infty} c_m e^{2\pi i m t}, \qquad c_m = \int_0^1 F(t) e^{-2\pi i m t} dt.$$

(For a twice-differentiable periodic function the coefficients are of size at most a constant over m^2, so the series converges absolutely and uniformly, and its sum is continuous with the same coefficients as F, hence equal to F.) Compute c_m by inserting the series for F and integrating term by term (uniform convergence):

$$\begin{aligned} c_m = \sum_n \int_0^1 f(t+n) e^{-2\pi i m t} dt &= \sum_n \int_n^{n+1} f(u) e^{-2\pi i m (u-n)} du \\ &= \sum_n \int_n^{n+1} f(u) e^{-2\pi i m u} du = \int_{-\infty}^{\infty} f(u) e^{-2\pi i m u} du, \end{aligned}$$

where the second equality substitutes $u = t+n$, the third uses $e^{2\pi i m n} = 1$ for whole m, n, and the fourth joins the integrals over the adjacent unit intervals into one over the whole line. The last integral is the Fourier transform $\hat{f}(m)$. Setting $t = 0$ in the Fourier series, and noting $F(0) = \sum_n f(n)$,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \hat{f}(m). \tag{103}$$

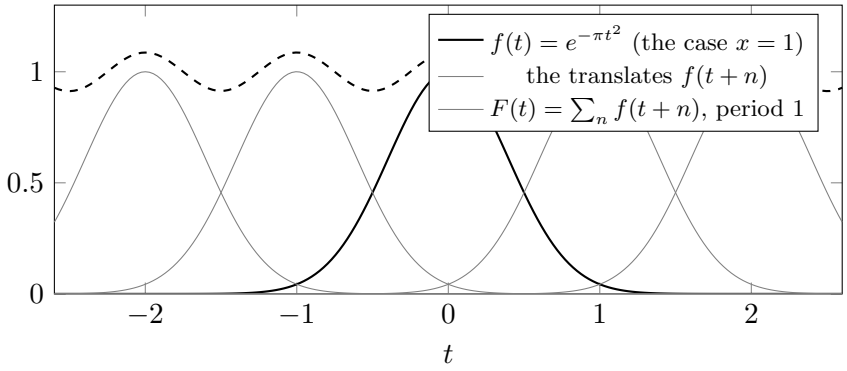


Figure 17: Poisson's summation formula, Step (a) of Section 14. The bell $f(t) = e^{-\pi x t^2}$ (drawn for $x = 1$) is copied to every whole-number position (grey); the sum of the copies, $F(t)$ (dashed), repeats with period 1 and so equals its Fourier series. Its value at $t = 0$ is $\sum_n f(n) = \theta(x)$; its Fourier coefficients are the transform $\hat{f}(m)$ at the whole numbers; equating the two is (103).

Step (b): the Fourier transform of the Gaussian. Let $g(\xi) = \int_{-\infty}^{\infty} e^{-\pi x t^2} e^{-2\pi i \xi t} dt$ for real ξ. Differentiate with respect to ξ under the integral sign (permitted because the derivative of the integrand, $-2\pi i t\, e^{-\pi x t^2} e^{-2\pi i \xi t}$, is bounded in modulus by $2\pi |t| e^{-\pi x t^2}$, which is integrable and independent of ξ):

$$g'(\xi) = \int_{-\infty}^{\infty} (-2\pi i t)\, e^{-\pi x t^2} e^{-2\pi i \xi t} dt.$$

Observe that $\frac{d}{dt}e^{-\pi x t^2} = -2\pi x t\, e^{-\pi x t^2}$, so $-2\pi i t\, e^{-\pi x t^2} = \frac{i}{x}\frac{d}{dt}e^{-\pi x t^2}$. Substitute and integrate by parts, differentiating $e^{-2\pi i\xi t}$ and integrating the derivative; the boundary term $\frac{i}{x}e^{-\pi x t^2}e^{-2\pi i\xi t}$ vanishes at $\pm\infty$:

$$g'(\xi) = \frac{i}{x}\int_{-\infty}^{\infty}\frac{d}{dt}\big(e^{-\pi x t^2}\big)e^{-2\pi i\xi t}dt = -\frac{i}{x}\int_{-\infty}^{\infty} e^{-\pi x t^2}\cdot(-2\pi i\xi)e^{-2\pi i\xi t}dt = \frac{i\cdot 2\pi i\xi}{x}\,g(\xi) = -\frac{2\pi\xi}{x}\,g(\xi),$$

using $i\cdot i=-1$. The differential equation $g' = -\frac{2\pi\xi}{x}g$ has the solution $g(\xi) = g(0)e^{-\pi\xi^2/x}$ (check: the derivative of $e^{-\pi\xi^2/x}$ is $-\frac{2\pi\xi}{x}e^{-\pi\xi^2/x}$), and $g(0) = \int e^{-\pi x t^2}dt = x^{-1/2}$ by the Gaussian integral. So

$$\hat f(\xi) = x^{-1/2}e^{-\pi\xi^2/x}. \tag{104}$$

Combining. Put (104) into (103):

$$\theta(x) = \sum_n e^{-\pi x n^2} = \sum_n f(n) = \sum_m \hat f(m) = x^{-1/2}\sum_m e^{-\pi m^2/x} = x^{-1/2}\theta(1/x).$$

So $\theta(x) = x^{-1/2}\theta(1/x)$, which is (102). ■

What the identity does. Solve (102) for $\psi(x)$: $\psi(x) = x^{-1/2}\psi(1/x) + \frac12 x^{-1/2} - \frac12$. As $x\to 0$, $1/x\to\infty$ and $\psi(1/x)\le 1.05e^{-\pi/x}$ is negligible, so $\psi(x) = \frac12 x^{-1/2} - \frac12 +$ (a term of size $x^{-1/2}e^{-\pi/x}$). The identity converts the behaviour of ψ near 0, which the series does not display, into the behaviour of ψ near ∞, which the series displays as exponential decay. Solved the other way, with $u = 1/x$: $\psi(1/u) = u^{1/2}\psi(u) + \frac12 u^{1/2} - \frac12$, the form used next.

14.5 Splitting the integral at $x = 1$ and transforming the piece near 0

Write the integral in (101) as the sum of its pieces over $(0,1]$ and $[1,\infty)$. In the piece over $(0,1]$ substitute $x = 1/u$: as x runs from 0 to 1, u runs from ∞ down to 1; $dx = -du/u^2$; and $x^{s/2-1} = u^{-(s/2-1)} = u^{1-s/2}$. Reversing the limits absorbs the minus sign:

$$\int_0^1 \psi(x)\,x^{s/2-1}dx = \int_1^\infty \psi(1/u)\,u^{1-s/2}\,\frac{du}{u^2} = \int_1^\infty \psi(1/u)\,u^{-s/2-1}\,du, \tag{105}$$

the exponents combining as $1 - s/2 - 2 = -s/2 - 1$. Now insert $\psi(1/u) = u^{1/2}\psi(u) + \frac12 u^{1/2} - \frac12$ and multiply out, using $u^{1/2}\cdot u^{-s/2-1} = u^{-(s+1)/2}$:

$$\int_1^\infty \psi(1/u)\,u^{-s/2-1}du = \int_1^\infty \psi(u)\,u^{-(s+1)/2}du + \frac12\int_1^\infty u^{-(s+1)/2}du - \frac12\int_1^\infty u^{-s/2-1}du. \tag{106}$$

The two elementary integrals, for $\operatorname{Re} s > 1$: the antiderivative of $u^{-(s+1)/2}$ is $u^{(1-s)/2}/\big(\frac{1-s}{2}\big)$, and $u^{(1-s)/2}\to 0$ as $u\to\infty$ because its exponent has negative real part, so the integral from 1 to ∞ is $(0-1)/\big(\frac{1-s}{2}\big) = \frac{2}{s-1}$; the antiderivative of $u^{-s/2-1}$ is $u^{-s/2}/(-\frac{s}{2})$, so the integral is $(0-1)/(-\frac{s}{2}) = \frac{2}{s}$. Hence the last two terms of (106) are $\frac12\cdot\frac{2}{s-1} - \frac12\cdot\frac{2}{s} = \frac{1}{s-1} - \frac1s = \frac{s-(s-1)}{s(s-1)} = \frac{1}{s(s-1)}$. Collecting the piece over $[1,\infty)$ of (101), the first term of (106) (with u renamed x), and this value:

$$\pi^{-s/2}\,\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \psi(x)\Big(x^{s/2-1} + x^{-(s+1)/2}\Big)dx \quad (\operatorname{Re} s > 1). \tag{107}$$

This is Riemann's equation. He writes the intermediate step as

$$\int_1^\infty \psi(x)x^{s/2-1}dx + \int_0^1 \psi(1/x)x^{(s-3)/2}dx + \tfrac12\int_0^1 \big(x^{(s-3)/2} - x^{s/2-1}\big)dx,$$

which is what one gets by applying Jacobi's identity to $\psi(x)$ on $(0,1)$ before substituting; the two forms differ only by the substitution $x\mapsto 1/x$ in the middle integral.

14.6 The right side of (107) is defined for every s

The integral over $[1,\infty)$ is an entire function of s, by the Morera argument: for $x \geq 1$, $\psi(x) \leq 1.05e^{-\pi x}$, and for s in a disc on which $|\sigma| \leq A$ the two powers of x have modulus at most $x^{A/2}$ each, so the integrand is bounded by $2.1\, e^{-\pi x}x^{A/2}$, integrable and independent of s, while for fixed x the integrand is entire in s. The term $1/(s(s-1))$ is a rational function with simple poles at $s=0$ and $s=1$. So the right side of (107) is meromorphic on the whole plane and, by Fact 2, equals $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ wherever the latter is defined. Dividing by $\pi^{-s/2}\Gamma(s/2)$, which is nonzero everywhere (Section 9), gives $\zeta(s)$ for every $s \neq 1$ a second time. The two constructions, this one and that of Section 11, define the same function, because Fact 2 leaves them no alternative.

14.7 What the second form shows at a glance

(i) *The functional equation, again.* The right side of (107) is unchanged when s is replaced by $1-s$: the term $1/(s(s-1))$ becomes $1/((1-s)(-s)) = 1/(s(s-1))$; the exponent $s/2-1$ becomes $(1-s)/2-1 = -(s+1)/2$, and the exponent $-(s+1)/2$ becomes $-((1-s)+1)/2 = -(2-s)/2 = s/2-1$, so the two powers of x are exchanged and their sum is unchanged. Hence $\Lambda(1-s) = \Lambda(s)$ with no computation at all: this is Riemann's second proof of the functional equation.

(ii) *The pole at $s=1$.* Near $s=1$ the right side is $\frac{1}{s-1}\cdot\frac{1}{s}$ + (finite), with $\frac{1}{s} \to 1$; and $\pi^{-1/2}\Gamma(\frac{1}{2}) = \pi^{-1/2}\sqrt{\pi} = 1$; so $\zeta(s)$ behaves like $1/(s-1)$: a simple pole with residue 1, as in Section 12.

(iii) *The value at $s=0$.* Near $s=0$ the right side is $\frac{1}{s}\cdot\frac{1}{s-1}$ + (finite), with $\frac{1}{s-1} \to -1$, so it behaves like $-1/s$; and $\Gamma(s/2)$ behaves like $2/s$, because $\Gamma(u) = \Gamma(u+1)/u$ with $u = s/2$ gives $\Gamma(s/2) \approx \Gamma(1)/(s/2) = 2/s$. Dividing, $\zeta(0) = \lim(-1/s)/(2/s) = -\frac{1}{2}$, as in Section 12.

(iv) *The trivial zeros.* At $s=-2k$ the function $\Gamma(s/2)$ has a pole (Section 9), so $1/\Gamma(s/2) = 0$, while the right side of (107) is finite; so $\zeta(-2k) = 0$.

(v) *No zeros on the real segment $0<s<1$.* For $0<s<1$ the alternating series $\eta(s) = 1-2^{-s}+3^{-s}-\cdots$ converges with terms decreasing in modulus, so its sum lies between the first partial sum 1 and the second $1-2^{-s}$, both positive; and the factor $1-2^{1-s}$ is negative because $1-s>0$ makes $2^{1-s}>1$. By (41), now valid everywhere, $\zeta(s) = \eta(s)/(1-2^{1-s}) < 0$ on $(0,1)$. The zeros in the strip, if there are any, are therefore off the real axis, and the next section is where Riemann looks for them.

15 Step six: the function $\xi(t)$, its two integral representations, and the zeros

Riemann now removes the pole and the trivial zeros by multiplying by $s(s-1)/2$, and turns the symmetry about $s=\frac{1}{2}$ into evenness in a new variable t. His words: "I now set $s = \frac{1}{2}+ti$ and $\Pi(s/2)(s-1)\pi^{-s/2}\zeta(s) = \xi(t)$, so that $\xi(t) = \frac{1}{2} - (tt+\frac{1}{4})\int_1^\infty \psi(x)\, x^{-3/4}\cos(\frac{1}{2}t\log x)\,dx$, or also $\xi(t) = 4\int_1^\infty \frac{d(x^{3/2}\psi'(x))}{dx}\, x^{-1/4}\cos(\frac{1}{2}t\log x)\,dx$. This function is finite for all finite values of t, and allows itself to be developed in powers of tt as a very rapidly converging series." In this section t is the new variable, related to s by $s = \frac{1}{2}+it$, so that $t = (s-\frac{1}{2})/i = -i(s-\frac{1}{2})$; real t corresponds to s on the line $\operatorname{Re} s = \frac{1}{2}$.

15.1 The first representation

Riemann's $\Pi(s/2)$ is $\Gamma(s/2+1)$, which by the recursion (57) equals $(s/2)\Gamma(s/2)$. So his definition reads

$$\xi = \frac{s}{2}\,\Gamma\Big(\frac{s}{2}\Big)(s-1)\,\pi^{-s/2}\zeta(s) = \frac{s(s-1)}{2}\cdot\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s). \tag{108}$$

Multiply both sides of (107) by $s(s-1)/2$. The term $1/(s(s-1))$ becomes $\frac{1}{2}$, and

$$\xi = \frac{1}{2} + \frac{s(s-1)}{2}\int_1^\infty \psi(x)\Big(x^{s/2-1} + x^{-(s+1)/2}\Big)\,dx. \tag{109}$$

Now put $s = \frac12 + it$. First, $s - 1 = -\frac12 + it$, so

$$s(s-1) = \Big(\frac12 + it\Big)\Big(-\frac12 + it\Big) = (it)^2 - \Big(\frac12\Big)^2 = -t^2 - \frac14 = -\Big(t^2 + \frac14\Big),$$

by the identity $(a+b)(-a+b) = b^2 - a^2$ with $a = \frac12$, $b = it$. Second, the exponents: $\frac{s}{2} - 1 = \frac14 + \frac{it}{2} - 1 = -\frac34 + \frac{it}{2}$ and $-\frac{s+1}{2} = -\frac{3/2+it}{2} = -\frac34 - \frac{it}{2}$. So the two powers of x are $x^{-3/4}x^{it/2}$ and $x^{-3/4}x^{-it/2}$, and since $x^{\pm it/2} = e^{\pm i(t\log x)/2}$, their sum is

$$x^{-3/4}\Big(e^{i\frac{t\log x}{2}} + e^{-i\frac{t\log x}{2}}\Big) = 2x^{-3/4}\cos\Big(\frac12 t\log x\Big).$$

Put both into (109); the factor 2 cancels the denominator 2:

$$\xi(t) = \frac12 - \Big(t^2 + \frac14\Big)\int_1^\infty \psi(x)\, x^{-3/4}\cos\Big(\frac12 t\log x\Big)dx, \tag{110}$$

which is Riemann's first formula.

Three properties read off (110)**.** First, $\xi(t)$ is an entire function of t: for $x \ge 1$, $\psi(x) \le 1.05e^{-\pi x}$, and for t in a disc with $|\operatorname{Im} t| \le T$ the cosine has modulus at most $e^{T\log x/2} = x^{T/2}$ (since $|\cos(a+ib)| \le e^{|b|}$), so the integrand is bounded by $1.05e^{-\pi x}x^{T/2-3/4}$, integrable and independent of t, and the Morera argument applies. Second, ξ is even, $\xi(-t) = \xi(t)$, because t^2 and cos are even in t; this is the symmetry $\Lambda(1-s) = \Lambda(s)$ written in the variable t, since replacing s by $1-s$ replaces $\frac12 + it$ by $\frac12 - it$. Third, $\xi(t)$ is real for real t, because every quantity in (110) is then real. The consequence of the last two: if t_0 is a zero of ξ, so are $-t_0$, $\bar t_0$ and $-\bar t_0$ (the conjugate of a zero of a function that is real on the real axis is a zero); so zeros come in groups of four unless t_0 is real or purely imaginary.

15.2 Which zeros ξ has

The factors by which ξ differs from ζ in (108) are $s(s-1)/2$, $\pi^{-s/2}$ and $\Gamma(s/2)$. The factor $s-1$ cancels the pole of ζ at $s = 1$, leaving the finite value $(s-1)\zeta(s) \to 1$. The factor s cancels the pole of $\Gamma(s/2)$ at $s = 0$, since $s\Gamma(s/2) = 2\cdot\frac{s}{2}\Gamma(\frac{s}{2}) = 2\Gamma(\frac{s}{2}+1) \to 2\Gamma(1) = 2$. At $s = 0$ and at $s = 1$ (the points $t = \pm i/2$) the factor $s(s-1)$ in (109) vanishes, so $\xi = \frac12$ at both. The poles of $\Gamma(s/2)$ at $s = -2, -4, \dots$ cancel the trivial zeros of ζ (each is a simple pole against a simple zero, product finite and, as one checks from (109), nonzero). Neither $\Gamma(s/2)$ nor $\pi^{-s/2}$ has a zero anywhere. Therefore the zeros of $\xi(t)$ are exactly the zeros of $\zeta(s)$ in the strip $0 \le \operatorname{Re} s \le 1$, each written in the variable $t = -i(s - \frac12)$. A zero of ζ at $s = \frac12 + it$ with real t is a real zero of ξ; a zero of ζ off the line $\operatorname{Re} s = \frac12$ is a zero of ξ off the real axis. This is the dictionary in which Riemann states his hypothesis.

15.3 The second representation, by two integrations by parts

The second formula replaces ψ by the derivative of $x^{3/2}\psi'(x)$ and absorbs the factor $t^2 + \frac14$. Write $a = \frac{s}{2}$ and $b = \frac{1-s}{2}$. Then $a + b = \frac12$, so $a - \frac12 = -b$ and $b - \frac12 = -a$; and $s(s-1) = (2a)(2a-1) = (2a)(-2b) = -4ab$, since $2a - 1 = s - 1 = -(1-s) = -2b$. In these letters (109) reads

$$\xi = \frac12 - 2ab\int_1^\infty \psi(x)\big(x^{a-1} + x^{b-1}\big)dx. \tag{111}$$

Define

$$J_a = \int_1^\infty \frac{d}{dx}\Big(x^{3/2}\psi'(x)\Big)\, x^{a-1/2}\, dx. \tag{112}$$

Integrate by parts, taking $\frac{d}{dx}(x^{3/2}\psi'(x))$ as the factor to integrate (its antiderivative is $x^{3/2}\psi'(x)$) and $x^{a-1/2}$ as the factor to differentiate (derivative $(a-\frac12)x^{a-3/2}$):

$$J_a = \Big[x^{3/2}\psi'(x)\, x^{a-1/2}\Big]_{x=1}^{x=\infty} - \Big(a - \frac12\Big)\int_1^\infty x^{3/2}\psi'(x)\, x^{a-3/2}\, dx.$$

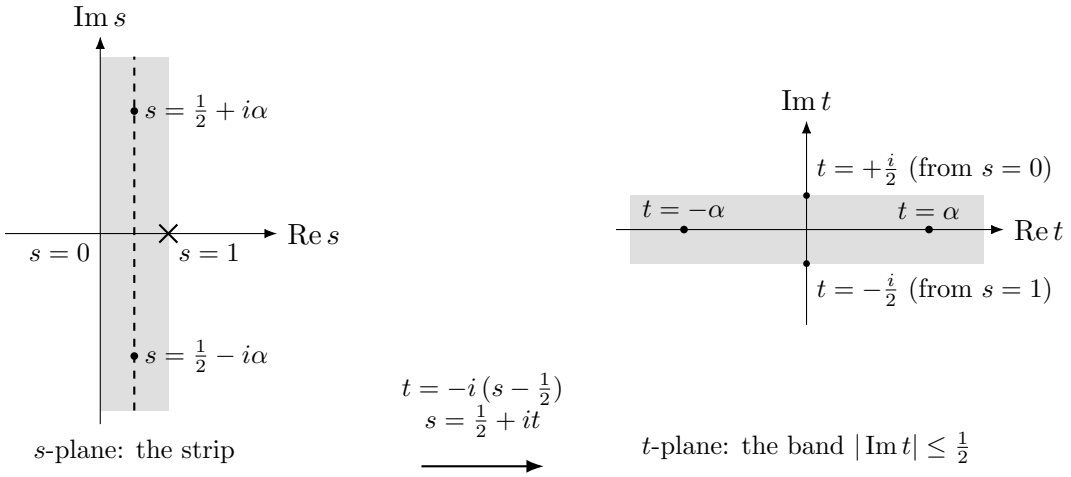


Figure 18: Riemann's change of variable $s = \frac{1}{2} + it$ in Section 15. The strip $0 \le \operatorname{Re} s \le 1$ (left, shaded) becomes the horizontal band $|\operatorname{Im} t| \le \frac{1}{2}$ (right, shaded); the critical line $\operatorname{Re} s = \frac{1}{2}$ (dashed) becomes the real t-axis; a pair of zeros $\frac{1}{2} \pm i\alpha$ on the line becomes the pair of real numbers $\pm\alpha$; the symmetry $s \mapsto 1 - s$ becomes $t \mapsto -t$; the points $s = 1$ and $s = 0$ become $t = \mp\frac{i}{2}$, where $\xi = \frac{1}{2}$. The hypothesis "all roots of $\xi(t)$ are real" is the statement that every zero in the strip sits on the dashed line.

The boundary term at $x = \infty$ is 0 because $\psi'(x) = -\pi \sum n^2 e^{-n^2\pi x}$ decays like $e^{-\pi x}$; at $x = 1$ it is $1 \cdot \psi'(1) \cdot 1 = \psi'(1)$, entering with a minus sign. In the remaining integral $x^{3/2} x^{a-3/2} = x^a$. Using $a - \frac{1}{2} = -b$,

$$J_a = -\psi'(1) + b \int_1^\infty \psi'(x)\, x^a \, dx. \tag{113}$$

Integrate by parts a second time, integrating ψ' (antiderivative ψ) and differentiating x^a (derivative ax^{a-1}):

$$\int_1^\infty \psi'(x)\, x^a \, dx = \Big[\psi(x) x^a\Big]_1^\infty - a \int_1^\infty \psi(x)\, x^{a-1}\, dx = -\psi(1) - a \int_1^\infty \psi(x)\, x^{a-1}\, dx,$$

the boundary term vanishing at ∞ (ψ decays like $e^{-\pi x}$) and equalling $\psi(1)$ at 1. Put this into (113):

$$J_a = -\psi'(1) - b\,\psi(1) - ab \int_1^\infty \psi(x)\, x^{a-1}\, dx. \tag{114}$$

The same computation with the roles of a and b exchanged (using $b - \frac{1}{2} = -a$) gives

$$J_b = -\psi'(1) - a\,\psi(1) - ab \int_1^\infty \psi(x)\, x^{b-1}\, dx. \tag{115}$$

Add (114) and (115), using $a + b = \frac{1}{2}$ in the $\psi(1)$ terms, and multiply by 2:

$$2\,(J_a + J_b) = -4\psi'(1) - \psi(1) - 2ab \int_1^\infty \psi(x)\big(x^{a-1} + x^{b-1}\big) dx. \tag{116}$$

The endpoint identity. The constant $-4\psi'(1) - \psi(1)$ is evaluated from Jacobi's identity. Let $\varphi(x) = 2\psi(x) + 1$; then (102) says $\varphi(x) = x^{-1/2}\varphi(1/x)$ for all $x > 0$. Differentiate both sides with respect to x, using the product rule on the right and the chain rule for $\varphi(1/x)$, whose derivative is $\varphi'(1/x) \cdot (-1/x^2)$:

$$\varphi'(x) = -\frac{1}{2} x^{-3/2} \varphi(1/x) + x^{-1/2} \varphi'(1/x) \cdot \Big(-\frac{1}{x^2}\Big).$$

Set $x = 1$, where $1/x = 1$: $\varphi'(1) = -\frac{1}{2}\varphi(1) - \varphi'(1)$, so $2\varphi'(1) = -\frac{1}{2}\varphi(1)$, that is, $\varphi'(1) = -\frac{1}{4}\varphi(1)$. Since $\varphi' = 2\psi'$ and $\varphi(1) = 2\psi(1) + 1$, this is $2\psi'(1) = -\frac{1}{4}(2\psi(1) + 1)$, and multiplying by 2,

$$4\psi'(1) + \psi(1) + \frac{1}{2} = 0, \qquad \text{so} \qquad -4\psi'(1) - \psi(1) = \frac{1}{2}. \tag{117}$$

Assembling. Put (117) into (116): $2(J_a+J_b)=\frac12-2ab\int_1^\infty\psi(x)(x^{a-1}+x^{b-1})dx$, which by (111) is ξ. Finally express J_a+J_b in the variable t: with $s=\frac12+it$, $a-\frac12=\frac{s}{2}-\frac12=-\frac14+\frac{it}{2}$ and $b-\frac12=-\frac14-\frac{it}{2}$, so $x^{a-1/2}+x^{b-1/2}=x^{-1/4}\big(x^{it/2}+x^{-it/2}\big)=2x^{-1/4}\cos(\frac12 t\log x)$, and

$$\xi(t)=2\,(J_a+J_b)=4\int_1^\infty\frac{d}{dx}\Big(x^{3/2}\psi'(x)\Big)\,x^{-1/4}\cos\Big(\frac12 t\log x\Big)dx, \tag{118}$$

which is Riemann's second formula. What it adds to the first: $\xi(t)$ is displayed as a cosine transform of one real function of x, with the factor $t^2+\frac14$ absorbed. Expanding $\cos(\frac12 t\log x)=\sum_{k\ge0}(-1)^k\frac{(t\log x)^{2k}}{2^{2k}(2k)!}$ and integrating term by term gives $\xi(t)=\sum_{k\ge0}c_k t^{2k}$, a power series in t^2 with real coefficients c_k, each an integral against $(\log x)^{2k}$ times an exponentially small weight, hence rapidly decreasing. This is the "very rapidly converging series" of Riemann's sentence; it is what makes $\xi(t)$ computable for real t and lets the zeros be located numerically.

15.4 The count of zeros, and the hypothesis

Riemann then states, with an indication of proof but not a proof, that the number of zeros of $\xi(t)$ whose real part lies between 0 and T is approximately

$$N(T)\approx\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}, \tag{119}$$

the relative error tending to 0 as T grows. His indication: integrate $d\log\xi(t)$ around the boundary of the rectangle $0\le\operatorname{Re}t\le T$, $-\frac12\le\operatorname{Im}t\le\frac12$ (in the variable s, the part of the strip $0\le\operatorname{Re}s\le1$ with $0\le\operatorname{Im}s\le T$); by the argument principle (a consequence of the residue theorem applied to ξ'/ξ, whose residue at a simple zero is 1) this integral is $2\pi i$ times the number of zeros inside, and the factor $\Gamma(s/2)$ in ξ supplies the main term (119) through Stirling's formula, $\log\Gamma(z)=(z-\frac12)\log z-z+\frac12\log(2\pi)+(\text{small})$. Hans von Mangoldt (1854–1925) proved (119) in 1895, thirty-six years after the paper, with an error of size at most a constant times $\log T$.

Riemann continues: "One finds in fact about this many real roots within these bounds, and it is very likely that all of the roots are real." A real root t of ξ is a zero of ζ at $s=\frac12+it$, on the line $\operatorname{Re}s=\frac12$. The sentence is the Riemann hypothesis: every zero of ζ in the strip lies on that line. He adds that he set the question aside after some fleeting attempts because it was not needed for the goal of his paper, which was the formula for the number of primes. Nothing in Sections 5 to 15 decides it.

16 What Riemann did next with the continued function, and what was proved later

The continuation was not Riemann's goal; it was the instrument. The remaining four pages of the paper use ζ on the whole plane to write the number of primes below x as an exact sum over the zeros. The steps are listed here with their status, because three of them were asserted by Riemann and proved by others, all of whom worked after him (the table of Section 3 gives the dates). In this section x is a real number greater than 1, p a prime, n, m, k positive whole numbers.

16.1 The product over the zeros: what "determined by its zeros" means

The statement for polynomials, which is the model. A polynomial $Q(t)$ of degree d with leading coefficient c and roots $\alpha_1,\dots,\alpha_d$ (each listed as often as its multiplicity) factors as

$$Q(t)=c\,(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_d), \tag{120}$$

by the fundamental theorem of algebra. Two consequences. First, the list of roots fixes Q except for the one number c: if Q_1 and Q_2 have the same roots with the same multiplicities, then Q_1/Q_2

is the constant c_1/c_2. That is the exact meaning of "determined by its zeros up to a constant." Second, if $Q(0) \neq 0$, dividing each factor by $-\alpha_j$ rewrites (120) as

$$Q(t) = Q(0) \prod_{j=1}^{d} \Big(1 - \frac{t}{\alpha_j}\Big), \tag{121}$$

since both sides are polynomials with the same roots and the same value at $t = 0$. If moreover Q is even, its roots come in pairs $\pm\alpha$, and the two factors of a pair multiply to $1 - t^2/\alpha^2$.

Why the statement is not automatic for an entire function. For a function analytic on the whole plane the list of zeros does *not* in general fix the function up to a constant: e^t has no zeros at all and is not constant, and $e^t Q(t)$ has exactly the zeros of $Q(t)$. So any factorisation of an entire function over its zeros must carry an extra factor $e^{g(t)}$ with g entire, and some further hypothesis is needed to pin g down. The hypothesis is a bound on growth. Riemann's ξ satisfies, by (110) and Stirling's formula, $|\xi(t)| \leq e^{C|t|\log|t|}$ for large $|t|$ and some constant C (so ξ grows no faster than $e^{|t|^{1+\varepsilon}}$ for every $\varepsilon > 0$; one says ξ has order 1).

The statement for ξ. Riemann asserts, and Hadamard proved in 1893, that

$$\xi(t) = \xi(0) \prod_{\alpha} \Big(1 - \frac{t^2}{\alpha^2}\Big), \tag{122}$$

the product running over the zeros α of ξ with positive real part, each counted with its multiplicity, and each paired with its partner $-\alpha$ (which is why only t^2 appears, exactly as for an even polynomial). The constant $\xi(0)$ is the value of (110) at $t = 0$, namely $\frac{1}{2} - \frac{1}{4}\int_1^\infty \psi(x) x^{-3/4} dx = 0.4971\ldots$. For t not a zero, taking logarithms gives

$$\log \xi(t) = \log \xi(0) + \sum_{\alpha} \log\Big(1 - \frac{t^2}{\alpha^2}\Big). \tag{123}$$

In words: the function ξ, and with it ζ, is completely fixed once the zeros α are known; there is no freedom beyond the one number $\xi(0)$, which is itself fixed by (110). This is the precise sense in which the zeros carry all the information about the primes: $J(x)$ is recovered from $\log\zeta$ by Theorem 16.4 below, $\log\zeta$ is $\log\xi$ plus elementary terms by (133), and $\log\xi$ is the sum over the zeros (123).

What has to be proved, and who proved it. Three things. (i) The product in (122) converges: this needs $\sum_\alpha 1/|\alpha|^2 < \infty$, which follows from the count $N(T) \approx \frac{T}{2\pi}\log\frac{T}{2\pi}$ of Section 15, since then the zeros with $|\alpha|$ between T and $2T$ number about $T\log T$ and contribute about $T\log T/T^2 = \log T/T$ to the sum, which is summable over $T = 1, 2, 4, 8, \ldots$. (ii) The quotient $\xi(t)\big/\prod_\alpha(1 - t^2/\alpha^2)$ is entire with no zeros, so it equals $e^{g(t)}$ for some entire g. (iii) The growth bound on ξ, together with a lower bound for the product away from the zeros, forces g to be a polynomial of degree at most 1; evenness of both ξ and the product forces the degree-1 term to vanish; so e^g is a constant, and setting $t = 0$ shows the constant is $\xi(0)$. Riemann's text indicates step (iii) by comparing the growth of $\log\xi$ with $N(T)$. Step (i) rests on the zero count, which Riemann stated and von Mangoldt proved in 1895. Steps (ii) and (iii) in full are the theory of entire functions of finite order, which Jacques Hadamard (1865–1963, born six years after the paper appeared) built in 1893 for exactly this purpose. Nothing in this paper proves (i)–(iii); what this paper does is state exactly what they assert, and use (122) in the next subsections only where Riemann used it.

16.2 The logarithm of $\zeta(s)$ for complex s, defined so that it is single-valued

For a complex number the logarithm has many values (item 16 of Section 4), so "$\log\zeta(s)$" must be defined before it is used. For $\operatorname{Re} s > 1$ define

$$L(s) = \sum_{p} \sum_{n \geq 1} \frac{1}{n}\, p^{-ns}, \tag{124}$$

the sum over all primes p and all exponents $n = 1, 2, 3, \ldots$. It converges absolutely: $|p^{-ns}| = p^{-n\sigma}$ by (52), and $\sum_p \sum_n p^{-n\sigma}/n \le \sum_p \sum_n p^{-n\sigma} = \sum_p p^{-\sigma}/(1-p^{-\sigma}) \le 2\sum_p p^{-\sigma} \le 2\zeta(\sigma)$, finite for $\sigma > 1$. Each term is an entire function of s, the convergence is uniform on every set $\sigma \ge 1+\delta$ (Weierstrass's test with the bound just written at $\sigma = 1+\delta$), so L is analytic on $\operatorname{Re} s > 1$ by the Morera argument of Section 8.

Claim 16.1. *$e^{L(s)} = \zeta(s)$ for $\operatorname{Re} s > 1$. So $L(s)$ is a logarithm of $\zeta(s)$; for real $s > 1$ it is the ordinary real logarithm of Section 5.5; and it is the one logarithm of ζ that is analytic on the half-plane and real on the real axis. It is written $\log\zeta(s)$ from here on.*

Proof. First, for a complex number u with $|u| < 1$, the series $\sum_{k\ge1} u^k/k$ converges absolutely, and $\exp\big(\sum_{k\ge1} u^k/k\big) = 1/(1-u)$. Both sides are analytic functions of u on the disc $|u| < 1$; for real u with $0 \le u < 1$ they are equal by (31), since the exponential of $-\log(1-u)$ is $1/(1-u)$; the real segment has limit points in the disc; so by Fact 2 they are equal on the whole disc. Apply this with $u = p^{-s}$, which has $|u| = p^{-\sigma} < 1$: $\exp\big(\sum_n p^{-ns}/n\big) = 1/(1-p^{-s})$. For the primes up to P, the exponential of a sum is the product of the exponentials, so

$$\exp\Big(\sum_{p\le P}\sum_{n\ge1}\frac{p^{-ns}}{n}\Big) = \prod_{p\le P}\exp\Big(\sum_{n\ge1}\frac{p^{-ns}}{n}\Big) = \prod_{p\le P}\frac{1}{1-p^{-s}}.$$

Let $P \to \infty$. The exponent on the left tends to $L(s)$, and exp is continuous, so the left side tends to $e^{L(s)}$. The right side tends to $\zeta(s)$ by the product formula for complex s, (53). Hence $e^{L(s)} = \zeta(s)$. □

16.3 The prime powers, and the step function $J(x)$

The double series (124) is indexed by pairs (p, n). Each pair corresponds to one whole number $m = p^n$, and conversely a number that is a power of a prime determines its prime and its exponent uniquely (by unique factorisation, Section 6). So the series can be indexed by the prime powers themselves. For every whole number $m \ge 1$ define the weight

$$a_m = \begin{cases} \dfrac{1}{n} & \text{if } m = p^n \text{ for some prime } p \text{ and some } n \ge 1, \\ 0 & \text{otherwise (including } m = 1). \end{cases} \tag{125}$$

Thus $a_2 = a_3 = a_5 = a_7 = 1$, $a_4 = a_9 = a_{25} = \frac12$, $a_8 = a_{27} = \frac13$, $a_{16} = \frac14$, and $a_6 = a_{10} = a_{12} = 0$. Then

$$\log\zeta(s) = \sum_{m\ge2} a_m\, m^{-s} \qquad (\operatorname{Re} s > 1), \tag{126}$$

which is (124) with its terms relabelled. Now define, for real $x \ge 1$, the step function

$$J(x) = \sum_{m\le x} a_m = \sum_{p^n\le x}\frac{1}{n}, \tag{127}$$

the sum of the weights of all prime powers not exceeding x (Figure 9). Three properties follow directly from the definition. $J(x) = 0$ for $x < 2$, since 2 is the smallest prime power. J is constant on each interval between consecutive prime powers and jumps upward by a_m at $x = m$. And $J(x) \le \sum_{m\le x} 1 \le x$, since every weight is at most 1. Riemann writes $f(x)$ for this function.

J in terms of the prime-counting function. Group the prime powers $p^n \le x$ by their exponent n. Those with exponent n are the numbers p^n with $p \le x^{1/n}$ (n-th root of x), and there are $\pi(x^{1/n})$ of them, each of weight $1/n$; the groups for different n are disjoint and together exhaust the prime powers; and the group is empty once $x^{1/n} < 2$, that is, once $n > \log x/\log 2$. Hence

$$J(x) = \pi(x) + \frac12\pi\big(x^{1/2}\big) + \frac13\pi\big(x^{1/3}\big) + \cdots = \sum_{n\ge1}\frac1n\,\pi\big(x^{1/n}\big), \tag{128}$$

a finite sum. Check at $x = 10$: $\pi(10) = 4$ (the primes $2, 3, 5, 7$), $\pi(\sqrt{10}) = \pi(3.16) = 2$, $\pi(10^{1/3}) = \pi(2.15) = 1$, $\pi(10^{1/4}) = \pi(1.78) = 0$; so $J(10) = 4 + \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 1 = 5\frac{1}{3}$, which agrees with the direct count ($2, 3, 5, 7$ with weight 1; $4, 9$ with weight $\frac{1}{2}$; 8 with weight $\frac{1}{3}$) and with Figure 9.

16.4 One term as an integral, and the whole series as one integral

Lemma 16.2. *For* $\operatorname{Re} s > 0$ *and any real* $m \geq 1$,

$$m^{-s} = s \int_m^\infty x^{-s-1}\, dx = s \int_1^\infty [x \geq m]\, x^{-s-1}\, dx, \tag{129}$$

where $[x \geq m]$ *denotes the step function equal to* 1 *when* $x \geq m$ *and* 0 *when* $x < m$ *(Figure 19).*

Proof. The function $x^{-s}/(-s)$ has derivative x^{-s-1}. Its value at $x = m$ is $-m^{-s}/s$, and as $x \to \infty$ it tends to 0 because $|x^{-s}| = x^{-\sigma}$ with $\sigma > 0$. So $\int_m^\infty x^{-s-1}dx = 0 - (-m^{-s}/s) = m^{-s}/s$, and multiplying by s gives the first equality. The second is a rewriting: inserting the factor $[x \geq m]$ cuts the integral over $[1, \infty)$ down to the integral over $[m, \infty)$. □

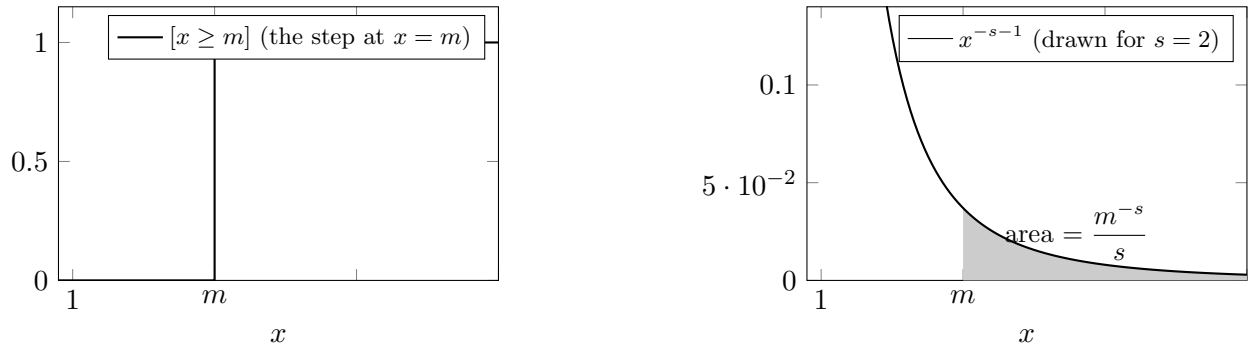


Figure 19: The identity (129) behind the integral form of $\log \zeta$. Left: the step function $[x \geq m]$, equal to 0 for $x < m$ and 1 for $x \geq m$; the function $J(x)$ of Figure 9 is the sum of such steps, one of height $1/n$ at each prime power $m = p^n$. Right: the shaded area under x^{-s-1} from $x = m$ onward equals m^{-s}/s, so s times the integral of $[x \geq m]\, x^{-s-1}$ over $x \geq 1$ is exactly the term m^{-s}. Adding over the prime powers with their weights turns the series for $\log \zeta(s)$ into $s \int_1^\infty J(x) x^{-s-1} dx$.

Now multiply (129) by a_m and add over the whole numbers m from 2 to M, for a fixed cut-off M; a finite sum of integrals is the integral of the sum:

$$\sum_{m=2}^{M} a_m m^{-s} = s \int_1^\infty \Big(\sum_{m=2}^{M} a_m [x \geq m] \Big) x^{-s-1}\, dx = s \int_1^\infty J_M(x)\, x^{-s-1}\, dx, \tag{130}$$

where $J_M(x) = \sum_{m \leq \min(x, M)} a_m$ is the step function J with its jumps beyond M removed: $J_M(x) = J(x)$ for $x \leq M$ and $J_M(x) = J(M)$ for $x > M$. Let $M \to \infty$. The left side of (130) tends to $\log \zeta(s)$ by (126). On the right, the difference $J(x) - J_M(x)$ is 0 for $x \leq M$ and lies between 0 and $J(x) \leq x$ for $x > M$, so

$$\Big| s \int_1^\infty \big(J(x) - J_M(x) \big) x^{-s-1} dx \Big| \leq |s| \int_M^\infty x \cdot x^{-\sigma-1} dx = |s| \int_M^\infty x^{-\sigma} dx = \frac{|s|\, M^{1-\sigma}}{\sigma - 1},$$

by the antiderivative of (8) with M as the lower limit, and this tends to 0 because $\sigma > 1$. Therefore

$$\log \zeta(s) = s \int_1^\infty J(x)\, x^{-s-1}\, dx \qquad (\operatorname{Re} s > 1), \tag{131}$$

which Riemann writes as $\frac{1}{s} \log \zeta(s) = \int_1^\infty f(x) x^{-s-1} dx$. The primes are present in this integral in one way only: as the positions of the jumps of J. The integral itself is an operation of the calculus on a function of the continuous variable x.

16.5 The inversion: Fourier's theorem in the form Riemann needs, proved

Equation (131) expresses $\log\zeta$ through J. Riemann now reverses it, obtaining J through $\log\zeta$, "by Fourier's theorem". The theorem of Fourier (1811, 1822) in question is the inversion formula of item 26 in Section 4; Riemann had worked over it in his 1854 Habilitation thesis (Section 2). The connection is the substitution $x = e^u$: then $x^{-s-1}dx = e^{-su}du$, and (131) reads $\log\zeta(c+i\omega)/(c+i\omega) = \int_0^\infty \big(J(e^u)e^{-cu}\big)e^{-i\omega u}du$, which says that $\log\zeta(c+i\omega)/(c+i\omega)$ is the Fourier transform of the function $u \mapsto J(e^u)e^{-cu}$; Fourier's inversion formula then recovers that function. Rather than quote the general theorem, the statement needed is proved here directly, by a residue computation; the lemma is now named after Perron (1908), but it is what "Fourier's theorem" amounts to for a step function.

Lemma 16.3 (Perron's discontinuous integral)**.** *Let $y > 0$, $c > 0$ and $T > 0$ be real, and put*

$$I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds, \qquad y^s = e^{s\log y},$$

the integral along the vertical segment from $c - iT$ to $c + iT$ (Figure 20). Then

$$\textit{(i) if } y>1: \quad \Big|I(y,T) - 1\Big| \le \frac{y^c}{\pi T \log y};$$
$$\textit{(ii) if } y<1: \quad \big|I(y,T)\big| \le \frac{y^c}{\pi T\,|\log y|};$$
$$\textit{(iii) if } y=1: \quad \Big|I(1,T) - \tfrac12\Big| \le \frac{c}{\pi T}.$$

So as $T \to \infty$, $I(y,T)$ tends to 1, $\frac12$ or 0 according as $y > 1$, $y = 1$ or $y < 1$.

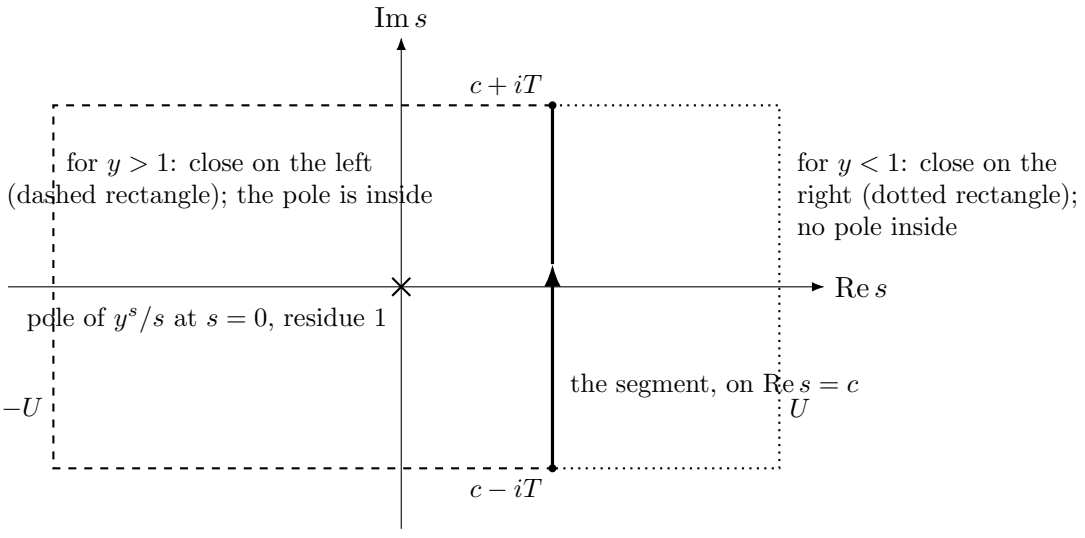


Figure 20: Perron's integral, Lemma 16.3. The integral of y^s/s is taken up the vertical segment from $c-iT$ to $c+iT$ (thick). For $y > 1$ the segment is closed into a rectangle on the left (dashed); the rectangle contains the pole at $s = 0$, whose residue is $y^0 = 1$, and the three added sides contribute little because $|y^s| = y^{\mathrm{Re}\,s}$ is small far to the left. For $y < 1$ it is closed on the right (dotted), where $|y^s|$ is small, and there is no pole, so the integral is nearly 0.

Proof. (i) Let $y > 1$ and take $U > 0$. Consider the rectangle with corners $c - iT$, $c + iT$, $-U + iT$, $-U - iT$, run counterclockwise: up the right side from $c-iT$ to $c+iT$, leftward along the top from $c + iT$ to $-U + iT$, down the left side, rightward along the bottom back to $c - iT$. The function y^s/s is analytic everywhere except at $s = 0$, a simple pole with residue $\lim_{s\to0} s\cdot y^s/s = y^0 = 1$, and

$s = 0$ lies inside the rectangle because $-U < 0 < c$. By the residue theorem the integral around the rectangle is $2\pi i \cdot 1$. So

$$I(y,T) = 1 - \frac{1}{2\pi i}\Big(\int_{\text{top}} + \int_{\text{left}} + \int_{\text{bottom}}\Big).$$

On the top side, $s = \sigma + iT$ with σ running from c down to $-U$; there $|y^s| = y^\sigma$ and $|s| \geq T$, so the modulus of the integral is at most $\frac{1}{T}\int_{-U}^{c} y^\sigma d\sigma \leq \frac{1}{T}\int_{-\infty}^{c} y^\sigma d\sigma = \frac{y^c}{T\log y}$, the last integral being $[y^\sigma/\log y]_{-\infty}^{c}$ with $y^\sigma \to 0$ as $\sigma \to -\infty$ because $y > 1$. The bottom side gives the same bound. On the left side, $s = -U + it$ with $|t| \leq T$: $|y^s| = y^{-U}$, $|s| \geq U$, length $2T$, so its integral has modulus at most $2Ty^{-U}/U$, which tends to 0 as $U \to \infty$. Letting $U \to \infty$ and collecting, $|I(y,T) - 1| \leq \frac{1}{2\pi} \cdot \frac{2y^c}{T\log y} = \frac{y^c}{\pi T \log y}$.

(ii) Let $y < 1$. Close the segment with a rectangle on the right, with corners $c \pm iT$ and $U \pm iT$. The pole at $s = 0$ lies to the left of the line $\operatorname{Re} s = c > 0$, so there is no singularity inside, and by Cauchy's theorem the integral around the rectangle is 0. On the right side $|y^s| = y^U \to 0$ as $U \to \infty$ (because $y < 1$), and on the top and bottom $\frac{1}{T}\int_c^\infty y^\sigma d\sigma = \frac{y^c}{T|\log y|}$. The same bookkeeping gives $|I(y,T)| \leq \frac{y^c}{\pi T|\log y|}$.

(iii) Let $y = 1$. Then $y^s = 1$, and with $s = c + it$, $ds = i\,dt$,

$$I(1,T) = \frac{1}{2\pi i}\int_{-T}^{T} \frac{i\,dt}{c+it} = \frac{1}{2\pi}\int_{-T}^{T} \frac{c - it}{c^2+t^2}\,dt = \frac{1}{2\pi}\int_{-T}^{T} \frac{c\,dt}{c^2+t^2} = \frac{1}{\pi}\arctan\frac{T}{c},$$

the odd part $-it/(c^2+t^2)$ integrating to 0 over the symmetric interval. Since $\frac{\pi}{2} - \arctan\frac{T}{c} = \arctan\frac{c}{T} \leq \frac{c}{T}$, this gives $|I(1,T) - \frac{1}{2}| \leq \frac{c}{\pi T}$. □

Theorem 16.4. *Let $c > 1$ and let $x > 1$ be a real number that is not a prime power. Then*

$$J(x) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\log\zeta(s)}{s}\,x^s\,ds. \tag{132}$$

If x is a prime power, the limit is $J(x) - \frac{1}{2}a_x$, the average of the values of J just to the left and just to the right of x.

Proof. On the segment from $c - iT$ to $c + iT$ one has $|m^{-s}| = m^{-c}$ and $|x^s/s| \leq x^c/c$ (since $|s| \geq \operatorname{Re} s = c$). So the terms of the series $\log\zeta(s)x^s/s = \sum_m a_m m^{-s}x^s/s$ are bounded on the segment by the constants $a_m m^{-c}x^c/c$, whose sum is $\log\zeta(c)\,x^c/c$, finite; by Weierstrass's test the series converges uniformly on the segment and may be integrated term by term:

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\log\zeta(s)}{s}x^s ds = \sum_{m\geq 2} a_m \cdot \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{(x/m)^s}{s} ds = \sum_{m\geq 2} a_m\, I\Big(\frac{x}{m}, T\Big),$$

using $m^{-s}x^s = (x/m)^s$. Write $\delta(y) = 1$, $\frac{1}{2}$ or 0 according as $y > 1$, $y = 1$ or $y < 1$. Since x is not a prime power, every m with $a_m \neq 0$ has $m \neq x$, and $\sum_m a_m\delta(x/m) = \sum_{m<x} a_m = J(x)$. Subtracting,

$$\Big|\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\log\zeta(s)}{s}x^s ds - J(x)\Big| \leq \sum_{m\geq 2} a_m\Big|I\Big(\frac{x}{m},T\Big) - \delta\Big(\frac{x}{m}\Big)\Big| \leq \sum_{m\geq 2} a_m \frac{(x/m)^c}{\pi T\,|\log(x/m)|},$$

by cases (i) and (ii) of the lemma. Let η be the smallest value of $|\log(x/m)|$ over the prime powers m; it is positive, because x is not a prime power, the prime powers nearest to x on either side give a positive value, and the farther ones give larger values. Then the right side is at most $\frac{x^c}{\pi T\eta}\sum_m a_m m^{-c} = \frac{x^c\log\zeta(c)}{\pi T\eta}$, which tends to 0 as $T \to \infty$. For x a prime power the one term with $m = x$ is handled by case (iii), contributing $a_x \cdot \frac{1}{2}$ in place of $a_x \cdot 1$. □

Equation (132) is Riemann's $f(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\log\zeta(s)}{s}x^s ds$, his a being the c above, and his remark that at a jump the integral gives the average of the two sides is the last sentence of the theorem. So far everything is proved: $J(x)$, a function whose jumps are the prime powers, is an integral of $\log\zeta$ along a vertical line in the complex s-plane lying to the right of $\operatorname{Re} s = 1$.

16.6 From the inversion integral to the explicit formula: what Riemann did, and what is proved

To evaluate (132), Riemann needs $\log\zeta(s)$ not just on the line $\operatorname{Re} s = c$ but everywhere the line might be moved to, and in a form that separates its singularities. He obtains that form from the definition of ξ and the product over the zeros. From (108), $\zeta(s) = \dfrac{2\,\pi^{s/2}}{s(s-1)\Gamma(s/2)}\,\xi$, so, taking logarithms,

$$\log\zeta(s) = \log\xi - \log(s-1) - \log s + \log 2 + \frac{s}{2}\log\pi - \log\Gamma\Big(\frac{s}{2}\Big). \tag{133}$$

Each term on the right is a known function with known singularities: $-\log(s-1)$ is singular at the pole $s=1$ of ζ; $\log\xi = \log\xi(0) + \sum_\alpha \log(1 - t^2/\alpha^2)$ by (122), with $t^2 = -(s-\frac12)^2$, is singular exactly at the zeros $\rho = \frac12 \pm i\alpha$ in the strip; $\log\Gamma(s/2)$ is singular at $s = 0, -2, -4, \ldots$, the trivial zeros; and the rest are elementary. Riemann substitutes (133) into (132) and evaluates the integral term by term. The pattern of each evaluation is the same and is worth seeing on the one term that matters most. Consider the logarithmic integral $\operatorname{Li}(x) = \int_0^x dt/\log t$ of item 24 (a function of x; the letters abbreviate "logarithmic integral"). Its derivative is $\operatorname{Li}'(x) = 1/\log x$, and more generally, for a complex constant β, the chain rule gives

$$\frac{d}{dx}\operatorname{Li}\big(x^\beta\big) = \frac{1}{\log(x^\beta)}\cdot\beta x^{\beta-1} = \frac{\beta x^{\beta-1}}{\beta\log x} = \frac{x^{\beta-1}}{\log x}. \tag{134}$$

Transform this derivative as in (131), for $\operatorname{Re} s > \operatorname{Re}\beta$, and then differentiate with respect to s under the integral sign:

$$\frac{d}{ds}\int_1^\infty \frac{x^{\beta-1}}{\log x}\,x^{-s}\,dx = \int_1^\infty \frac{x^{\beta-1}}{\log x}\cdot(-\log x)\,x^{-s}\,dx = -\int_1^\infty x^{\beta-s-1}dx = -\frac{1}{s-\beta},$$

by Lemma 16.2 with $m=1$ and $s-\beta$ in place of s. And $-\frac{1}{s-\beta}$ is the s-derivative of $-\log(s-\beta)$. So, up to a constant that the differentiation removes, the transform of $\frac{d}{dx}\operatorname{Li}(x^\beta)$ is $-\log(s-\beta)$: a logarithmic singularity of $\log\zeta$ at the point $s=\beta$ corresponds, under the inversion, to the function $\operatorname{Li}(x^\beta)$. Applied to (133): the term $-\log(s-1)$ gives $+\operatorname{Li}(x)$; each factor $\log(1-s/\rho) = \log(\rho - s) - \log\rho$ in $\log\xi$ gives $-\operatorname{Li}(x^\rho)$; the singularities of $-\log\Gamma(s/2)$ at the trivial zeros give, when their contributions are added, the small term $\int_x^\infty dt/\big(t(t^2-1)\log t\big)$; and the constants give $-\log 2$. The outcome is Riemann's explicit formula,

$$J(x) = \operatorname{Li}(x) - \sum_\rho \operatorname{Li}(x^\rho) + \int_x^\infty \frac{dt}{t\,(t^2-1)\log t} - \log 2, \tag{135}$$

where the sum runs over the zeros $\rho = \frac12 \pm i\alpha$ in the strip, each paired with its partner and the pairs taken in order of increasing α; Riemann writes the sum as $\sum_\alpha \big[\operatorname{Li}(x^{1/2+\alpha i}) + \operatorname{Li}(x^{1/2-\alpha i})\big]$, and the term $\operatorname{Li}(x^\rho)$ for complex ρ means $\int_0^{x^\rho} dt/\log t$ along a path from 0 to the complex number x^ρ.

Status. Everything up to and including Theorem 16.4 is proved above. The term-by-term evaluation that leads from (132) and (133) to (135) is carried out by Riemann at the level just shown: each type of term is matched with its Li by a derivative computation like (134), and the sum over the infinitely many zeros is taken in the order of increasing α without a proof that this order makes the sum converge to the right value. The two missing pieces are the product formula (122) for ξ, supplied by Hadamard in 1893, and the justification of the term-by-term passage with its order of summation, supplied by von Mangoldt in 1895, thirty-six years after the paper. With those two papers, (135) is a theorem.

16.7 Recovering $\pi(x)$ from $J(x)$: Möbius inversion, proved

Equation (128) gives J in terms of π; Riemann needs π in terms of J. The tool is the function $\mu(n)$ of August Ferdinand Möbius (1790–1868), defined in item 25 of Section 4: $\mu(1)=1$, $\mu(n)=(-1)^k$ if n is a product of k different primes, and $\mu(n)=0$ if a prime squared divides n. The function dates from 1832; Möbius was alive when Riemann wrote, though Riemann does not name him.

The key property. For every $k \geq 1$,

$$\sum_{d \mid k} \mu(d) = \begin{cases} 1 & (k=1), \\ 0 & (k>1), \end{cases} \tag{136}$$

the sum running over the positive divisors d of k. Proof: for $k=1$ the only divisor is 1 and $\mu(1)=1$. For $k>1$, let $p_1,\dots,p_r$ be the different primes dividing k, with $r \geq 1$. The divisors d with $\mu(d) \neq 0$ are the products of some subset of these primes (the empty subset giving $d=1$), and $\mu(d) = (-1)^{(\text{size of the subset})}$. A set of r elements has $\binom{r}{j}$ subsets of size j, so the sum is $\sum_{j=0}^{r} \binom{r}{j}(-1)^j = (1-1)^r = 0$ by the binomial theorem.

The inversion. Apply (128) at the point $x^{1/n}$ and multiply by $\mu(n)/n$:

$$\frac{\mu(n)}{n} J\big(x^{1/n}\big) = \sum_{m \geq 1} \frac{\mu(n)}{nm}\, \pi\big((x^{1/n})^{1/m}\big) = \sum_{m \geq 1} \frac{\mu(n)}{nm}\, \pi\big(x^{1/(nm)}\big).$$

Add over $n \geq 1$ (finitely many nonzero terms, since $\pi(x^{1/(nm)})=0$ once $nm > \log x/\log 2$):

$$\sum_{n \geq 1} \frac{\mu(n)}{n} J\big(x^{1/n}\big) = \sum_{n \geq 1} \sum_{m \geq 1} \frac{\mu(n)}{nm}\, \pi\big(x^{1/(nm)}\big).$$

Group the pairs (n,m) by the value $k=nm$ of their product: for a given k, the pairs are $(d, k/d)$ with d running over the divisors of k, and $\frac{\mu(n)}{nm} = \frac{\mu(d)}{k}$. So the double sum is $\sum_{k \geq 1} \frac{1}{k} \pi(x^{1/k}) \sum_{d \mid k} \mu(d)$, and by (136) only $k=1$ survives:

$$\pi(x) = \sum_{n \geq 1} \frac{\mu(n)}{n}\, J\big(x^{1/n}\big) = J(x) - \frac{1}{2} J\big(x^{1/2}\big) - \frac{1}{3} J\big(x^{1/3}\big) - \frac{1}{5} J\big(x^{1/5}\big) + \frac{1}{6} J\big(x^{1/6}\big) - \frac{1}{7} J\big(x^{1/7}\big) + \cdots, \tag{137}$$

the signs and omissions following $\mu(1)=1$, $\mu(2)=\mu(3)=\mu(5)=\mu(7)=-1$, $\mu(4)=0$, $\mu(6)=+1$. ■ This is the formula with which Riemann's paper ends: (135) gives J exactly in terms of the zeros, and (137) turns J into the number of primes.

16.8 What the formula says

The leading term $\mathrm{Li}(x)$ in (135) is the approximation to $\pi(x)$ that Gauss (1777–1855) had read off from tables of primes in 1792–93 and reported in a letter to Encke in 1849, ten years before Riemann's paper; Legendre's $x/(\log x - 1.08366)$, in the book Riemann read at sixteen, is a cruder version of the same thing. Every zero ρ contributes an oscillating correction $\mathrm{Li}(x^\rho)$ whose size is about $x^{\mathrm{Re}\,\rho}/\log x$. So the real parts of the zeros govern how far $\pi(x)$ can stray from $\mathrm{Li}(x)$: if every zero has $\mathrm{Re}\,\rho = \frac{1}{2}$, the total error is of order $\sqrt{x}\log x$, and (von Koch, 1901) it is of that order only if they do. Riemann also observes that the term $-\frac{1}{2}J(\sqrt{x}) \approx -\frac{1}{2}\mathrm{Li}(\sqrt{x})$ in (137) makes $\mathrm{Li}(x)$ exceed $\pi(x)$ by about $\sqrt{x}/\log x$ on average, which explains the excess that Gauss and others had seen in the tables. The prime number theorem, the statement that $\pi(x)/\mathrm{Li}(x) \to 1$, was proved in 1896, thirty-seven years after the paper, by Hadamard and by Charles-Jean de la Vallée Poussin (1866–1962), each by showing that $\zeta(1+it) \neq 0$ for real $t \neq 0$ and feeding that into the explicit formula. So the sequence of operations traced in Sections 5 to 14, from Euler's series to the continued function, is the first half of the proof that the primes have the density Gauss conjectured; the second half is a statement about where the zeros of the continued function are, and the sharpest form of that statement is the one Riemann left as a conjecture.

17 Functions that select the primes: what exists, what cannot, and the open problem of their cost

Section 6 said that the restriction in $\sum_p$ is a restriction of the index set and that no formula is used to produce the primes. That leaves a question standing: does a function exist that, applied to a whole number n, returns 1 when n is prime and 0 otherwise, and if so, in what forms, and at what cost? This section answers the parts that are settled and states precisely the part that is not. Throughout, n is a whole number, p a prime, and "cost" means the number of elementary operations needed to evaluate the function at n, measured against the number of digits of n, which is about $\log n / \log 10$.

17.1 The question, stated so that it has an answer

A function f on the whole numbers with $f(n) = 1$ exactly when n is prime is called here a *selector* of the primes. Any procedure that decides whether n is prime is a selector, and any selector is such a procedure; the two phrases name one thing. So the question "is there a selector that is not a test" has no content. The questions with content are: in which forms can a selector be written, and how cheaply can one be evaluated. The first is answered below in four forms; the second is the open problem at the end.

17.2 What cannot exist: a polynomial all of whose values are prime

Theorem 17.1 (Goldbach, letter to Euler of 1752; proof by Euler)**.** *Let $F(x_1, \dots, x_k)$ be a polynomial with whole-number coefficients. If F takes a prime value at every point $(a_1, \dots, a_k)$ with whole-number coordinates, then F is a constant.*

Proof. Pick any whole-number point $a = (a_1, \dots, a_k)$ and put $p = F(a)$, a prime. Fix one coordinate direction j and consider the one-variable polynomial $g(t) = F(a_1, \dots, a_j + t, \dots, a_k)$. Replacing t by a multiple mp of p changes each monomial of g by a multiple of p, so $g(mp) \equiv g(0) = p \equiv 0 \pmod p$ for every whole number m. By hypothesis $g(mp)$ is a prime, and a prime divisible by p is p itself. So the polynomial g takes the value p at the infinitely many points $t = 0, p, 2p, \dots$, and a polynomial that takes one value infinitely often is that constant; hence $g'(t)$ is the zero polynomial, that is, the partial derivative $\partial F / \partial x_j$ vanishes at a. Since a and j were arbitrary, every partial derivative of F vanishes at every whole-number point; a polynomial that vanishes at all whole-number points is the zero polynomial (for one variable, it has infinitely many roots; for more, fix all variables but one and use induction). So all partial derivatives of F are zero, and F is a constant. □

The consequence is that the primes cannot be the full set of values of any polynomial. What they can be is the set of *positive* values of a polynomial, which is the second form below.

17.3 Four forms in which a selector exists

Form 1: an elementary closed form, from Wilson's theorem.

Theorem 17.2 (Wilson's theorem; stated by Wilson in 1770, proved by Lagrange in 1771)**.** *A whole number $n \geq 2$ is prime if and only if n divides $(n-1)! + 1$.*

Proof. Let $n = p$ be prime. Every a in $1, \dots, p-1$ has an inverse a' in the same range with $aa' \equiv 1 \pmod p$ (since p and a have no common factor, some multiple of a leaves remainder 1 on division by p). The number a is its own inverse exactly when $a^2 \equiv 1$, that is, when p divides $(a-1)(a+1)$, that is, when $a = 1$ or $a = p-1$. So the numbers $2, \dots, p-2$ fall into pairs $\{a, a'\}$ with product $\equiv 1$, and $(p-1)! \equiv 1 \cdot (p-1) \equiv -1 \pmod p$, which says p divides $(p-1)! + 1$. Now let n be composite, $n = ab$ with $1 < a \leq b < n$. If $a \neq b$, both a and b occur among $1, \dots, n-1$, so n divides $(n-1)!$. If $a = b$ and $a > 2$, then a and $2a$ are different numbers below $n = a^2$ (because $2a < a^2$ when $a > 2$), and their product $2a^2$ is a multiple of n, so again n divides $(n-1)!$. The remaining case $n = 4$

has $3! = 6$, and 4 does not divide 7. So for composite n the number $(n-1)!+1$ leaves a nonzero remainder on division by n. $\square$

Since $(n-1)! = \Gamma(n)$ by Section 9, define for $n \geq 2$

$$\chi(n) = \Big\lfloor \cos^2\Big(\pi\,\frac{\Gamma(n)+1}{n}\Big)\Big\rfloor, \tag{138}$$

where $\lfloor y \rfloor$ is the largest whole number not exceeding y. If n is prime, $(\Gamma(n)+1)/n$ is a whole number m, $\cos(\pi m) = \pm 1$, $\cos^2 = 1$, and $\chi(n) = 1$. If n is composite, $(\Gamma(n)+1)/n = m + r/n$ with a remainder r satisfying $1 \leq r \leq n-1$, so $\cos^2(\pi m + \pi r/n) = \cos^2(\pi r/n) < 1$ and $\chi(n) = 0$. Thus χ is a selector. It is Willans's formula (1964). Every sum over primes in this paper can be rewritten with it:

$$\sum_p f(p) = \sum_{n\geq 2} \chi(n) f(n), \qquad \zeta(s) = \prod_{n\geq 2} \Big(1 - \frac{1}{n^s}\Big)^{-\chi(n)} \quad (\operatorname{Re} s > 1). \tag{139}$$

Its cost is that of computing $(n-1)!$: about n multiplications of numbers with up to $n \log n$ digits, which is exponential in the number of digits of n.

Form 2: a polynomial whose positive values are exactly the primes. Jones, Sato, Wada and Wiens (1976) wrote down an explicit polynomial $F(a, b, \ldots, z)$ with whole-number coefficients, of degree 25 in 26 variables, with the property that, as the variables run over all whole numbers, the set of *positive* values of F is exactly the set of primes. So n is prime if and only if there are whole numbers $a, \ldots, z$ with $F(a, \ldots, z) = n$. This does not contradict Theorem 17.1, because F also takes negative values, which are discarded. The construction rests on the theorem of Matiyasevich (1970), completing work of Davis, Putnam and Robinson, that every set of whole numbers that a machine can list is the set of positive values of some polynomial; the primes are such a set, and the four authors produced the polynomial explicitly. Matiyasevich (1977) reduced the number of variables to 10. The polynomial contains no test; the test is the search over the 26 variables, which is unbounded.

Form 3: a selector built from ζ itself. The third form uses the tools of Sections 8 and 16 and nothing else. Define the prime zeta function $P(s) = \sum_p p^{-s}$ for $\operatorname{Re} s > 1$; it converges absolutely because $\sum_p p^{-\sigma} \leq \zeta(\sigma)$.

Lemma 17.3 (Glaisher, 1891)**.** *For* $\operatorname{Re} s > 1$,

$$P(s) = \sum_{k\geq 1} \frac{\mu(k)}{k}\, \log \zeta(ks), \tag{140}$$

with $\log\zeta$ *the series* (124) *and* μ *the Möbius function of Section 16.*

Proof. By (124) with ks in place of s, $\log\zeta(ks) = \sum_p \sum_{n\geq 1} p^{-nks}/n$. Multiply by $\mu(k)/k$ and add over $k \geq 1$:

$$\sum_{k\geq 1} \frac{\mu(k)}{k} \log\zeta(ks) = \sum_{k\geq 1} \sum_p \sum_{n\geq 1} \mu(k)\, \frac{p^{-nks}}{nk}.$$

The triple series converges absolutely: $|\log\zeta(ks)| \leq \sum_p \sum_n p^{-nk\sigma} \leq 2\sum_p p^{-k\sigma} \leq 2\big(\zeta(k\sigma) - 1\big)$, and $\zeta(k\sigma) - 1 \leq 2 \cdot 2^{-k\sigma}$ for $k\sigma \geq 2$ by the comparison of Section 5.2, so the sum over k is dominated by a geometric series. It may therefore be rearranged. Group the pairs (n, k) by their product $m = nk$; for a fixed m the pairs are $(m/k, k)$ with k a divisor of m, and $\frac{\mu(k)}{nk} = \frac{\mu(k)}{m}$. So the triple series equals

$$\sum_p \sum_{m\geq 1} \frac{p^{-ms}}{m} \sum_{k\mid m} \mu(k) = \sum_p p^{-s},$$

because $\sum_{k\mid m} \mu(k)$ is 1 for $m = 1$ and 0 for $m > 1$ by (136). $\square$

Theorem 17.4. *For every whole number $n \geq 2$ and every real $c > 1$,*

$$\mathbf{1}_P(n) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} P(s)\, \frac{(n+\frac{1}{2})^s - (n-\frac{1}{2})^s}{s}\, ds, \tag{141}$$

where $\mathbf{1}_P(n)$ is 1 if n is prime and 0 otherwise. By Lemma 17.3, the right side is an expression in ζ alone.

Proof. On the segment from $c-iT$ to $c+iT$, $|p^{-s}| = p^{-c}$ and $|(n\pm\frac{1}{2})^s/s| \leq (n+\frac{1}{2})^c/c$, so the series $P(s)\big((n+\frac{1}{2})^s - (n-\frac{1}{2})^s\big)/s$ converges uniformly there by Weierstrass's test and may be integrated term by term. In the notation of Lemma 16.3, the integral equals

$$\sum_p \Big[I\Big(\frac{n+\frac{1}{2}}{p}, T\Big) - I\Big(\frac{n-\frac{1}{2}}{p}, T\Big)\Big].$$

The arguments $(n\pm\frac{1}{2})/p$ are never equal to 1, because p is a whole number and $n\pm\frac{1}{2}$ is not; so only cases (i) and (ii) of the lemma occur, and as $T\to\infty$ each bracket tends to $\delta\big((n+\frac{1}{2})/p\big) - \delta\big((n-\frac{1}{2})/p\big)$, which is 1 when $n-\frac{1}{2} < p < n+\frac{1}{2}$, that is, when $p = n$, and 0 otherwise. The sum over p of these limits is therefore 1 if n is prime and 0 if not. The difference between the integral and this limit is at most

$$\sum_p \frac{(n+\frac{1}{2})^c + (n-\frac{1}{2})^c}{\pi T\,\eta}\, p^{-c} \leq \frac{2(n+\frac{1}{2})^c\, P(c)}{\pi T\,\eta}, \qquad \eta = \min_p \Big|\log\frac{n\pm\frac{1}{2}}{p}\Big| > 0,$$

by the lemma, and this tends to 0 as $T\to\infty$. □

The theorem is the paper's own machinery turned into a selector: the integral counts the primes in the interval $(n-\frac{1}{2}, n+\frac{1}{2})$, which contains the whole number n and nothing else. Its cost is the cost of ζ, which presupposes the sieve of Section 5.4; so it is a test in the form of an integral.

Form 4: a selector of polynomial cost. Agrawal, Kayal and Saxena (2002) gave a selector whose cost is polynomial in the number of digits of n: it decides primality by testing whether the polynomial identity $(x+a)^n \equiv x^n + a \pmod{n,\ x^r-1}$ holds for a few small a and a suitable small r, a test that does no trial division at all. Their bound was about the twelfth power of the number of digits; Lenstra and Pomerance brought it to about the sixth power. This is the cheapest selector known.

17.4 The open problem

> **Open problem.** Let d be the number of digits of n. Agrawal, Kayal and Saxena show that a selector of cost about d^6 exists. The only lower bound known is the cost of reading n, about d. Determine the true cost: is there a selector of cost close to d, as there is for multiplying two d-digit numbers, or is there a proof that every selector costs more than a fixed power of d greater than 1?

Nothing in this paper bears on the answer, and nothing in Riemann's. What this paper does settle is the status of the four forms: a selector can be written as an elementary formula with a floor (Form 1), as the positive part of a polynomial (Form 2), as a contour integral of ζ (Form 3), and as a polynomial-cost algorithm (Form 4), while it cannot be written as a polynomial all of whose values are prime (Theorem 17.1). In Euler's and Riemann's work none of the four is used: the sum $\sum_p$ is defined by its index set, and the sieve identity of Section 5.4 carries the primes into the calculus and back without a selector ever being evaluated.

18 The sequence of operations in one table

Each row is one operation on one equation; the right-hand column locates it in Riemann's text (Wilkins translation).

Step	Equation operated on	Operation performed	Result	Valid for	Place in the 1859 paper
0 (Euler 1737)	$\zeta(s)=\sum n^{-s}$, real $s>1$	Sieve: multiply by $(1-p^{-s})$ prime by prime	$\zeta(s)=\prod(1-p^{-s})^{-1}$	$s>1$	Cited in the opening sentence
1	The series with s complex	Term-by-term bound $n^{-\sigma}$; Weierstrass; same sieve	ζ analytic, product holds, $\zeta\neq 0$	$\operatorname{Re} s>1$	Opening sentence
2	$\Gamma(s)=\int_0^\infty e^{-x}x^{s-1}dx$	Scale $x\mapsto nx$; sum over n; swap sum and integral	$\Gamma(s)\zeta(s)=\int_0^\infty x^{s-1}dx/(e^x-1)$	$\operatorname{Re} s>1$	First two displayed equations
3	The integral of step 2	Replace the path $(0,\infty)$ by the loop C around 0; fix the branch of $(-x)^{s-1}$ real on the negative axis	$2\sin(\pi s)\Gamma(s)\zeta(s)=i\int_C(-x)^{s-1}dx/(e^x-1)$; right side entire, so ζ is meromorphic on $\mathbb{C}$	all s	"If one now considers the integral…"
4	The loop formula at whole-number s	Rays cancel; residue at $x=0$ from the Bernoulli expansion	Pole at $s=1$ with residue 1; $\zeta(-n)=(-1)^nB_{n+1}/(n+1)$; $\zeta(-2k)=0$	whole numbers	"This equation now gives the value…"
5	The loop formula for $\operatorname{Re} s<0$	Expand the loop to the circle of radius $(2N+1)\pi$; residues at $x=\pm 2\pi in$; $N\to\infty$	$2\sin(\pi s)\Gamma(s)\zeta(s)=(2\pi)^s\cdot 2\sin(\pi s/2)\cdot\zeta(1-s)$, hence $\zeta(1-s)=2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s)$	all s, by continuation	"If the real part of s is negative…"
6	The functional equation of step 5	Legendre duplication and Euler reflection for Γ	$\pi^{-s/2}\Gamma(s/2)\zeta(s)$ unchanged under $s\mapsto 1-s$	all s	"known properties of the function Π"
7	$\Gamma(s/2)=\int_0^\infty e^{-u}u^{s/2-1}du$	Substitute $u=n^2\pi x$; sum over n	$\pi^{-s/2}\Gamma(s/2)\zeta(s)=\int_0^\infty\psi(x)x^{s/2-1}dx$	$\operatorname{Re} s>1$	"This property of the function induced me…"
8	The integral of step 7	Split at $x=1$; substitute $x\mapsto 1/x$ on $(0,1)$; apply Jacobi's $\theta(1/x)=\sqrt{x}\,\theta(x)$; evaluate two elementary integrals	$\pi^{-s/2}\Gamma(s/2)\zeta(s)=1/(s(s-1))+\int_1^\infty\psi(x)(x^{s/2-1}+x^{-(s+1)/2})dx$	all s	"and since $2\psi(x)+1=\dots$ (Jacobi, Fund. S. 184)"
9	The formula of step 8	Multiply by $s(s-1)/2$; set $s=\frac12+ti$	$\xi(t)=\frac12-(t^2+\frac14)\int_1^\infty\psi(x)\,x^{-3/4}\times\cos(\frac12 t\log x)\,dx$	all t	"I now set $s=\frac12+ti$…"
10	The first ξ formula	Two integrations by parts; endpoint identity $4\psi'(1)+\psi(1)+\frac12=0$ from Jacobi	$\xi(t)=4\int_1^\infty\frac{d}{dx}(x^{3/2}\psi'(x))\times x^{-1/4}\cos(\frac12 t\log x)\,dx$	all t	"or also…"

Steps 2 and 3 are the analytic continuation; step 5 is the first proof of the functional equation; steps 7 and 8 are the second construction of the continuation and the second proof of the functional equation; steps 9 and 10 prepare the study of the zeros. Nothing in steps 3 to 10 uses any property of ζ beyond the series and the product of steps 0 and 1; everything else is the Gamma function, Cauchy's theorem, and Jacobi's identity.

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